

# Moderate deviations for the volume of the Wiener sausage

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## Abstract

For  $a > 0$ , let  $W^a(t)$  be the  $a$ -neighbourhood of standard Brownian motion in  $\mathbb{R}^d$  starting at 0 and observed until time  $t$ . It is well-known that  $E|W^a(t)| \sim \kappa_a t$  ( $t \rightarrow \infty$ ) for  $d \geq 3$ , with  $\kappa_a$  the Newtonian capacity of the ball with radius  $a$ . We prove that

$$\lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P(|W^a(t)| \leq bt) = -I^{\kappa_a}(b) \in (-\infty, 0) \quad \text{for all } 0 < b < \kappa_a$$

and derive a variational representation for the rate function  $I^{\kappa_a}$ . We show that the optimal strategy to realise the above moderate deviation is for  $W^a(t)$  to ‘look like a Swiss cheese’:  $W^a(t)$  has random holes whose sizes are of order 1 and whose density varies on scale  $t^{1/d}$ . The optimal strategy is such that  $t^{-1/d}W^a(t)$  is delocalised in the limit as  $t \rightarrow \infty$ . This is markedly different from the optimal strategy for large deviations  $\{|W^a(t)| \leq f(t)\}$  with  $f(t) = o(t)$ , where  $W^a(t)$  is known to fill completely a ball of volume  $f(t)$  and nothing outside, so that  $W^a(t)$  has no holes and  $f(t)^{-1/d}W^a(t)$  is localised in the limit as  $t \rightarrow \infty$ .

We give a detailed analysis of the rate function  $I^{\kappa_a}$ , in particular, its behaviour near the boundary points of  $(0, \kappa_a)$  as well as certain monotonicity properties. It turns out that  $I^{\kappa_a}$  has an infinite slope at  $\kappa_a$  and, remarkably, for  $d \geq 5$  is nonanalytic at some critical point in  $(0, \kappa_a)$ , above which it follows a pure power law. This crossover is associated with a collapse transition in the optimal strategy.

We also derive the analogous moderate deviation result for  $d = 2$ . In this case  $E|W^a(t)| \sim 2\pi t / \log t$  ( $t \rightarrow \infty$ ), and we prove that

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \log P(|W^a(t)| \leq bt / \log t) = -I^{2\pi}(b) \in (-\infty, 0) \quad \text{for all } 0 < b < 2\pi.$$

The rate function  $I^{2\pi}$  has a finite slope at  $2\pi$ .

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### 1. Introduction and main results: Theorems 1–5 and Corollaries 1, 2

1.1 *The Wiener sausage.* Let  $\beta(t)$ ,  $t \geq 0$ , be the standard Brownian motion in  $\mathbb{R}^d$  – the Markov process with generator  $\Delta/2$  – starting at 0. Let  $P, E$  denote its probability law and expectation on path space. The Wiener sausage with radius  $a > 0$  is the process defined by

$$(1.1) \quad W^a(t) = \bigcup_{0 \leq s \leq t} B_a(\beta(s)), \quad t \geq 0,$$

where  $B_a(x)$  is the open ball with radius  $a$  around  $x \in \mathbb{R}^d$ . The Wiener sausage is an important mathematical object, because it is one of the simplest examples of a non-Markovian functional of Brownian motion. It plays a key role in the study of various stochastic phenomena, such as heat conduction and trapping in random media, as well as in the analysis of spectral properties of random Schrödinger operators.

A lot is known about the behaviour of the volume of  $W^a(t)$  as  $t \rightarrow \infty$ . For instance,

$$(1.2) \quad E|W^a(t)| \sim \begin{cases} \sqrt{8t/\pi} & (d=1) \\ 2\pi t/\log t & (d=2) \\ \kappa_a t & (d \geq 3), \end{cases}$$

with  $\kappa_a = a^{d-2} 2\pi^{d/2} / \Gamma(\frac{d-2}{2})$  the Newtonian capacity of  $B_a(0)$  associated with the Green function of  $(-\Delta/2)^{-1}$ , and

$$(1.3) \quad \text{Var}|W^a(t)| \asymp \begin{cases} t & (d=1) \\ t^2/\log^4 t & (d=2) \\ t \log t & (d=3) \\ t & (d \geq 4) \end{cases}$$

(Spitzer [21], Le Gall [17]). Moreover,  $|W^a(t)|$  satisfies the strong law and the central limit theorem for  $d \geq 2$ ; the limit law is Gaussian for  $d \geq 3$  and non-Gaussian for  $d = 2$  (Le Gall [18]). Note that for  $d \geq 2$  the Wiener sausage is a *sparse* object: since the Brownian motion typically travels a distance  $\sqrt{t}$  in each direction, (1.2) shows that most of the space in the convex hull of  $W^a(t)$  is not covered.

1.2. *Large deviations.* The large deviation properties of  $|W^a(t)|$  in the *downward* direction have been studied by Donsker and Varadhan [12], Bolthausen [6] and Sznitman [22]. For  $d \geq 2$  the outcome, proved in successive stages of refinement, reads as follows:

$$(1.4) \quad \lim_{t \rightarrow \infty} \frac{f(t)^{2/d}}{t} \log P(|W^a(t)| \leq f(t)) = -\frac{1}{2}\lambda_d$$

for any  $f: \mathbb{R}_+ \mapsto \mathbb{R}_+$  satisfying  $\lim_{t \rightarrow \infty} f(t) = \infty$  and

$$(1.5) \quad f(t) = \begin{cases} o(t/\log t) & (d=2) \\ o(t) & (d \geq 3), \end{cases}$$

where  $\lambda_d > 0$  is the smallest Dirichlet eigenvalue of  $-\Delta$  on the ball with unit volume. It turns out that the optimal strategy for the Brownian motion to realise the large deviation in (1.4) is to stay inside a ball with volume  $f(t)$  until time  $t$ , i.e., the Wiener sausage covers this ball entirely and nothing outside. (The optimality comes from the Faber-Krahn isoperimetric inequality, and the cost of staying inside the ball is  $\exp[-\frac{1}{2}\lambda_d t/f(t)^{2/d}]$  to leading order.) Thus, the optimal strategy is simple and  $f(t)^{-1/d}W^a(t)$  is *localised*. Note that, apparently, a large deviation *below* the scale of the mean ‘squeezes all the empty space out of the Wiener sausage’. Also note that the limit in (1.4) does not depend on  $a$ .

The law of the Brownian motion *conditioned* on the large deviation event  $\{|W^a(t)| \leq f(t)\}$  has been studied by Sznitman [23], Bolthausen [7] and Povel [20]. This law is indeed like the optimal strategy described above, with an explicitly known probability distribution for the centre of the ball the Brownian motion stays confined in.

1.3. *Moderate deviations.* The aim of the present paper is to extend (1.4), (1.5) by investigating deviations *on* the scale of the mean. We call such deviations moderate.<sup>1</sup> Our first main result reads:

**THEOREM 1.** *Let  $d \geq 3$  and  $a > 0$ . For every  $b > 0$ ,*

$$(1.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P(|W^a(t)| \leq bt) = -I^{\kappa_a}(b),$$

where

$$(1.7) \quad I^{\kappa_a}(b) = \inf_{\phi \in \Phi^{\kappa_a}(b)} \left[ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2(x) dx \right]$$

with

$$(1.8) \quad \Phi^{\kappa_a}(b) = \left\{ \phi \in H^1(\mathbb{R}^d): \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} \left( 1 - e^{-\kappa_a \phi^2(x)} \right) dx \leq b \right\}.$$

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<sup>1</sup>The term ‘moderate’ is often used for deviations away from the mean that are smaller than the scale of the mean, but in view of the contrast with (1.4), (1.5) we prefer this terminology.

The idea behind Theorem 1 is that the optimal strategy for the Brownian motion to realise the event  $\{|W^a(t)| \leq bt\}$  is to behave like a Brownian motion in a drift field  $xt^{1/d} \mapsto (\nabla\phi/\phi)(x)$  for some smooth  $\phi: \mathbb{R}^d \mapsto [0, \infty)$ . The cost of adopting this drift during a time  $t$  is the exponential of  $t^{(d-2)/d}$  times the integral in (1.7) to leading order. The effect of the drift is to push the Brownian motion towards the origin. Conditioned on adopting the drift, the Brownian motion spends time  $\phi^2(x)$  per unit volume in the neighbourhood of  $xt^{1/d}$ , and it turns out that the Wiener sausage covers a fraction  $1 - \exp[-\kappa_a \phi^2(x)]$  of the space in that neighbourhood. The best choice of the drift field is therefore given by a minimiser of the variational problem in (1.7), or by a minimising sequence.

We thus see that the optimal strategy for the Wiener sausage is to cover only part of the space and to leave *random holes*<sup>2</sup> whose sizes are of order 1 and whose density varies on scale  $t^{1/d}$ . This strategy is more complicated than for (1.4) and  $t^{-1/d}W^a(t)$  is *delocalised*. (In Section 5.1 it is shown that all minimisers or minimising sequences of (1.7) are strictly positive.) Note that, apparently, a moderate deviation *on* the scale of the mean ‘does not squeeze all the empty space out of the Wiener sausage’. Also note that the limit in (1.6) does depend on  $a$ .<sup>3</sup>

It is clear from (1.4), (1.5) that the case  $d = 2$  is critical. Our next main result is the following parallel of Theorem 1.

**THEOREM 2.** *Let  $d = 2$  and  $a > 0$ . For every  $b > 0$ ,*

$$(1.9) \quad \lim_{t \rightarrow \infty} \frac{1}{\log t} \log P(|W^a(t)| \leq bt / \log t) = -I^{2\pi}(b),$$

where  $I^{2\pi}(b)$  is given by the same formulas as in (1.7), (1.8), except that  $\kappa_a$  is replaced by  $2\pi$ .

Theorem 2 shows that for  $d = 2$  the moderate deviations have a polynomially small rather than an exponentially small probability. The optimal strategy is of the same type, but now the Wiener sausage lives on scale  $\sqrt{t/\log t}$ , which is only slightly below the diffusive scale. Contrary to the case  $d \geq 3$ , the rate function does not depend on  $a$ . This means that the random holes in the Swiss cheese have a typical size and a typical mutual distance that tend to infinity as  $t \rightarrow \infty$ , washing out the dependence on the radius of the Wiener sausage.

<sup>2</sup>The motto of this paper: ‘How a Wiener sausage turns into a Swiss cheese’.

<sup>3</sup>To prove that the law of the Brownian motion *conditioned* on the moderate deviation event  $\{|W^a(t)| \leq bt\}$  actually follows the optimal ‘Swiss cheese strategy’ requires substantial extra work. We shall not address this issue here. Even though we shall sometimes interpret our results in terms of this strategy, we have no pathwise statements to offer.

There is no result analogous to Theorems 1, 2 for  $d = 1$ , for the simple reason that the strong law fails (see (1.2), (1.3)). The variational problem in (1.7), (1.8) certainly continues to make sense for  $d = 1$ , but it does not describe the Wiener sausage: holes are impossible in  $d = 1$ .

**1.4. The rate function.** We proceed with a closer analysis of (1.7), (1.8). First we scale out the  $a$ -dependence and make some general statements about the rate function. Recall that  $\kappa_a \equiv 2\pi$  for  $d = 2$ .

**THEOREM 3.** *Let  $d \geq 2$  and  $a > 0$ ,*

(i) *For every  $b > 0$ ,*

$$(1.10) \quad I^{\kappa_a}(b) = \frac{1}{2\kappa_a^{2/d}} \chi(b/\kappa_a),$$

*where  $\chi: (0, \infty) \mapsto [0, \infty)$  is given by*

$$(1.11) \quad \chi(u) = \inf\{\|\nabla\psi\|_2^2: \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) \leq u\}.$$

(ii)  $\chi$  is continuous on  $(0, \infty)$ , strictly decreasing on  $(0, 1)$ , and equal to zero on  $[1, \infty)$ .

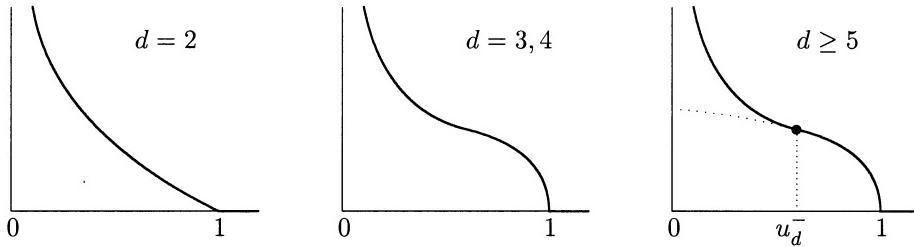
(iii)  $u \mapsto u^{2/d}\chi(u)$  is strictly decreasing on  $(0, 1)$  and

$$(1.12) \quad \lim_{u \downarrow 0} u^{2/d}\chi(u) = \lambda_d$$

with  $\lambda_d$  as defined below (1.5).

Theorem 3(iii) shows that the limit  $b \downarrow 0$  connects up nicely with (1.4), (1.5).

Our next two results show that the variational problem in (1.11) displays a surprising dimension dependence.



Qualitative picture of  $u \mapsto \chi(u)$  for

(i)  $d = 2$ ; (ii)  $d = 3, 4$ ; (iii)  $d \geq 5$ .

**THEOREM 4.** *Let  $2 \leq d \leq 4$ .*

(i) *For every  $u \in (0, 1)$  the variational problem in (1.11) has a minimiser that is strictly positive, radially symmetric (modulo shifts) and strictly decreasing in the radial component. Any other minimiser is of the same type.*

(ii)  $u \mapsto (1-u)^{-2/d} \chi(u)$  *is strictly decreasing on  $(0, 1)$  and*

$$(1.13) \quad \lim_{u \uparrow 1} (1-u)^{-2/d} \chi(u) = 2^{2/d} \mu_d,$$

*where*

$$(1.14) \quad \mu_d = \begin{cases} \inf\{\|\nabla \psi\|_2^2 : \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \|\psi\|_4 = 1\} & (d = 2, 3) \\ \inf\{\|\nabla \psi\|_2^2 : \psi \in D^1(\mathbb{R}^4), \|\psi\|_4 = 1\} & (d = 4) \end{cases}$$

*satisfying  $0 < \mu_d < \infty$ .*

**THEOREM 5.** *Let  $d \geq 5$ .*

(i) *Define*

$$(1.15) \quad \begin{aligned} \nu_d &= \inf\{\|\nabla \psi\|_2^2 : \psi \in D^1(\mathbb{R}^d), \int_{\mathbb{R}^d} (e^{-\psi^2} - 1 + \psi^2) = 1\} \\ \Sigma &= \text{the set of minimisers} \\ \Sigma^* &= \text{the set of local minimisers}. \end{aligned}$$

*Then  $0 < \nu_d < \infty$  and  $\emptyset \neq \Sigma^* \supseteq \Sigma$ . Moreover, all elements of  $\Sigma$  are strictly positive, radially symmetric (modulo shifts), strictly decreasing in the radial component, and there exists a constant  $K_d$  such that*

$$(1.16) \quad \begin{aligned} \|\psi\|_2^2 &> \frac{d}{d-2} \quad \text{for all } \psi \in \Sigma^*, \quad \|\psi\|_2^2 \leq K_d \quad \text{for all } \psi \in \Sigma. \end{aligned}$$

(ii) *Define  $2/d \leq u_d^* \leq u_d^- \leq u_d^+ \leq 1 - K_d^{-1} < 1$  by*

$$(1.17) \quad u_d^* = 1 - [\inf_{\psi \in \Sigma^*} \|\psi\|_2^2]^{-1}, \quad u_d^- = 1 - [\inf_{\psi \in \Sigma} \|\psi\|_2^2]^{-1}, \quad u_d^+ = 1 - [\sup_{\psi \in \Sigma} \|\psi\|_2^2]^{-1}.$$

*For every  $u \in (0, u_d^*]$  the variational problem in (1.11) has a minimiser that is strictly positive, radially symmetric (modulo shifts) and strictly decreasing in the radial component. Any other minimiser is of the same type. For every  $u \in (u_d^+, 1)$  the variational problem in (1.11) does not have a minimiser. There exists a minimising sequence  $(\psi_j)$  such that  $\psi_j(\cdot)$  converges weakly to  $\psi(\cdot(1-u)^{-1/d})$  in  $H^1(\mathbb{R}^d)$  as  $j \rightarrow \infty$  for some  $\psi \in \Sigma$ .*

(iii)  $u \mapsto (1-u)^{-(d-2)/d} \chi(u)$  *is strictly decreasing on  $(0, u_d^*]$ , nonincreasing and strictly greater than  $\nu_d$  on  $(u_d^+, u_d^-)$ , while*

$$(1.18) \quad (1-u)^{-(d-2)/d} \chi(u) \equiv \nu_d \quad \text{for } u \in [u_d^-, 1).$$

Note that  $\chi$  is nonanalytic at  $u_d^-$ . Whether or not (1.11) has a minimiser for  $u \in (u_d^*, u_d^+]$  and whether or not (1.15) has just one local minimiser both remain open. Possibly  $|\Sigma^*| = 1$ , in which case  $u_d^* = u_d^- = u_d^+ =: u_d$ , but this seems hard to settle (see §§5.6–5.8).

**1.5. Comments.** To explain the situation in Theorem 5, let us insert the scaling  $\psi(\cdot(1-u)^{-1/d})$  into (1.11) to obtain

$$(1.19) \quad (1-u)^{(d-2)/d}\chi(u) = \inf\{\|\nabla\psi\|_2^2: \psi \in H^1(\mathbb{R}^d), \|\psi\|_2^2 = (1-u)^{-1}, \\ \int_{\mathbb{R}^d}(e^{-\psi^2} - 1 + \psi^2) \geq 1\}.$$

In Section 5.1 it will be shown that the two constraints in (1.19) may be replaced by  $\|\psi\|_2^2 \leq (1-u)^{-1}$  and  $\int_{\mathbb{R}^d}(e^{-\psi^2} - 1 + \psi^2) = 1$ , after which we have a variational problem as in (1.15) but with an upper bound on  $\|\psi\|_2^2$ . Let us now consider the optimistic scenario where  $\Sigma^*$  has a unique element  $\psi^*$ . Then  $u_d^* = u_d^- = u_d^+ =: u_d$  and  $\|\psi^*\|_2^2 = (1-u_d)^{-1}$ . It turns out that for  $u \in (0, u_d]$  the variational problem in (1.19) has a minimiser because no  $L^2$ -mass wants to leak away to infinity (even though this minimiser has little to do with  $\psi^*$  itself). On the other hand, for  $u \in (u_d, 1)$  it has no minimiser, and any minimising sequence converges weakly to  $\psi^*$  by leaking  $L^2$ -mass. In the less optimistic scenario where  $|\Sigma^*| > 1$ , there is no leakage for  $u \in (0, u_d^*]$  and leakage for  $u \in (u_d^+, 1)$ .

The situation in Theorem 4 can be explained as follows. It turns out that for  $2 \leq d \leq 4$  all elements of  $\Sigma^*$  have infinite  $L^2$ -norm, so that  $u_d^* = u_d^- = u_d^+ = 1$ . Hence for any  $u \in (0, 1)$  there is no leakage and (1.19) has a minimiser.

The following points in Theorems 4 and 5 are noteworthy:

- I. At  $b = \kappa_a$  the rate function has an *infinite* slope for  $d \geq 3$  but a *finite* slope for  $d = 2$ .
- II. The scaling as  $b \uparrow \kappa_a$  is different for  $2 \leq d \leq 4$  and  $d \geq 5$ . Apparently a delicate dimension dependence is felt as the deviation becomes smaller than the mean. The fact that for  $d \geq 5$  there is no minimiser for  $u \in (u_d^+, 1)$  is to be interpreted as saying that the optimal strategy is *time-inhomogeneous* in the following sense. Let us again pretend that  $\Sigma^*$  has a unique element  $\psi^*$ , and let us put  $\rho(u) = (u - u_d)/(1 - u_d) \in (0, 1)$ . Then heuristically (recall footnote 3):
  - (1) Until time  $[1 - \rho(u)]t$  the Wiener sausage makes a Swiss cheese on scale  $t^{1/d}$  parametrised by  $\psi^*(\cdot(1-u)^{-1/d})$ , filling a volume  $\kappa_a[u - \rho(u)]t$ .
  - (2) After time  $[1 - \rho(u)]t$  it behaves like a typical Wiener sausage on scale  $\sqrt{t}$ , filling an additional volume  $\kappa_a\rho(u)t$ .

Thus, at time  $[1 - \rho(u)]t$  the optimal strategy undergoes a *collapse transition* from subdiffusive behaviour (scale  $t^{1/d}$ ) to diffusive behaviour (scale  $\sqrt{t}$ ). The picture is unclear when  $|\Sigma^*| > 1$  and  $u \in (u_d^*, u_d^+]$ . Still, we expect some type of collapse transition to occur.

III. For  $d \geq 3$ , the scaling of the rate function near  $\kappa_a$  does not connect up with the central limit theorem. Indeed, if we pick  $b = b_t$  with

$$(1.20) \quad b_t t = \begin{cases} \kappa_a t - c\sqrt{t \log t} & (d = 3) \\ \kappa_a t - ct & (d \geq 4) \end{cases}$$

for some  $c > 0$  and recall (1.2), (1.3), then we find from (1.10), (1.13) and (1.18) that

$$(1.21) \quad I^{\kappa_a}(b_t)t^{(d-2)/d} \rightarrow \infty \quad (t \rightarrow \infty)$$

instead of a finite limit. Therefore the moderate deviations are in a sense *anomalous*. For  $d = 2$ , on the other hand, we put

$$(1.22) \quad b_t \frac{t}{\log t} = \frac{2\pi t}{\log t} - c \frac{t}{\log^2 t}$$

for some  $c > 0$  and find that

$$(1.23) \quad \lim_{t \rightarrow \infty} I^{2\pi}(b_t) \log t \text{ exists in } (0, \infty).$$

So there is *no anomaly* in this case. Incidentally, for  $d \geq 3$  the correction term to the asymptotic mean is of smaller order than the asymptotic standard deviation, while for  $d = 2$  it is of the same order (Spitzer [21], Getoor [15]). For the above argument we may therefore indeed only consider the leading order terms given by (1.2), (1.3).

The anomaly for  $d \geq 3$  is somewhat surprising. It suggests that the central limit behaviour is controlled by the *local* fluctuations of the Wiener sausage, while the moderate and large deviations are controlled by the *global* fluctuations.

It remains open whether  $I^{2\pi}$  is convex for  $d = 2$  and whether  $I^{\kappa_a}$  has only one point of inflection for  $d \geq 3$ .

**1.6. Negative exponential moments.** We close this introduction with two corollaries. An immediate consequence of Theorem 1 is the following result.<sup>4</sup>

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<sup>4</sup>Sznitman [24, pp. 213–214] gives a heuristic derivation of Corollary 1 using his method of ‘enlargement of obstacles’.

COROLLARY 1. Let  $d \geq 3$  and  $a > 0$ . For every  $c > 0$ ,

$$(1.24) \quad \lim_{t \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log E \left( \exp[-ct^{-2/d}|W^a(t)|] \right) = -J^{\kappa_a}(c)$$

with

$$(1.25) \quad J^{\kappa_a}(c) = \inf_{b>0} [bc + I^{\kappa_a}(b)].$$

It follows from (1.25) that

$$(1.26) \quad J^{\kappa_a}(c) \begin{cases} = \kappa_a c & (0 < c \leq c_a^*) \\ < \kappa_a c & (c > c_a^*) \end{cases}$$

with

$$(1.27) \quad c_a^* = \max\{c > 0 : I^{\kappa_a}(b) \geq c(\kappa_a - b) \text{ for all } 0 < b \leq \kappa_a\}.$$

At  $c = c_a^*$ , the minimiser of (1.25) moves from  $b = \kappa_a$  to the interior of  $(0, \kappa_a]$ . Heuristically, this corresponds to a *collapse transition* in the optimal strategy for the Brownian motion associated with (1.24), (1.25), namely, from diffusive behaviour (scale  $\sqrt{t}$ ) to subdiffusive behaviour (scale  $t^{1/d}$ ). By Theorems 3(i), 4(ii) and 5(iii), the left derivative of  $I^{\kappa_a}$  at  $b = \kappa_a$  is  $-\infty$ . Therefore not only is  $c_a^* > 0$ , at  $c = c_a^*$  the minimiser of (1.25) is discontinuous. Heuristically, this means that the optimal strategy stays localised on scale  $t^{1/d}$  as  $c \downarrow c_a^*$ , i.e., the collapse transition is *first order*.

The analogue of Corollary 1 for  $d = 2$  follows from Theorem 2 and reads as follows.

COROLLARY 2. Let  $d = 2$  and  $a > 0$ . For every  $c > 0$ ,

$$(1.28) \quad \lim_{t \rightarrow \infty} \frac{1}{\log t} \log E \left( \exp[-ct^{-1} \log^2 t |W^a(t)|] \right) = -J^{2\pi}(c)$$

with

$$(1.29) \quad J^{2\pi}(c) = \inf_{b>0} [bc + I^{2\pi}(b)].$$

The same statements as in (1.26), (1.27) hold, again with  $c^* > 0$ , because by Theorem 4(ii) the left derivative of  $I^{2\pi}$  at  $b = 2\pi$  is strictly negative. Thus, also for  $d = 2$  there is a collapse transition. However,  $b \mapsto I^{2\pi}(b)/(2\pi - b)$  is strictly decreasing on  $(0, 2\pi)$  by Theorems 3(i) and 4(ii), and so at  $c = c^*$  the minimiser is continuous. This means that the optimal strategy does not stay localised on scale  $\sqrt{t/\log t}$  as  $c \downarrow c^*$ , i.e., the collapse transition is *second order*.

1.7. *Upward deviations.* Finally, the moderate and large deviations of  $|W^a(t)|$  in the *upward* direction are a complicated issue. Here the optimal strategy is entirely different from the previous ones, because the Wiener sausage tries to expand rather than to contract. Partial results have been obtained by van den Berg and Tóth [4] and van den Berg and Bolthausen [2].

## 2. The upper bound in Theorem 1

This section contains the main probabilistic part of the paper and, together with Sections 3 and 4, provides the proof of Theorems 1 and 2.

2.1. *Compactification: Propositions 1 and 2.* We begin by doing a standard compactification. Let  $\Lambda_N$  be the torus of size  $N > 0$ , i.e.,  $[-\frac{N}{2}, \frac{N}{2})^d$  with periodic boundary conditions. For  $t > 0$ , let  $\beta_{Nt^{1/d}}(s)$ ,  $s \geq 0$ , be the Brownian motion wrapped around  $\Lambda_{Nt^{1/d}}$ , and let  $W_{Nt^{1/d}}^a(s)$ ,  $s \geq 0$ , denote its Wiener sausage. Then trivially

$$(2.1) \quad P(|W^a(t)| \leq bt) \leq P(|W_{Nt^{1/d}}^a(t)| \leq bt)$$

for all  $a > 0$ ,  $b > 0$ ,  $N > 0$  and  $t > 0$ . Next, by Brownian scaling,  $|W_{Nt^{1/d}}^a(t)|$  has the same distribution as  $t|W_N^{at^{-1/d}}(t^{(d-2)/d})|$ . Hence, putting

$$(2.2) \quad \tau = t^{(d-2)/d}$$

we get

$$(2.3) \quad P(|W^a(t)| \leq bt) \leq P(|W_N^{a\tau^{-1/(d-2)}}(\tau)| \leq b).$$

The right-hand side of (2.3) involves the Wiener sausage on  $\Lambda_N$  at time  $\tau$  with a radius that shrinks with  $\tau$ .

In Sections 2.2–2.5 we shall prove the following:

**PROPOSITION 1.** *Let  $d \geq 3$  and  $a > 0$ . For every  $b > 0$  and  $N > 0$ ,*

$$(2.4) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|W_N^{a\tau^{-1/(d-2)}}(\tau)| \leq b) = -I_N^{\kappa_a}(b),$$

where  $I_N^{\kappa_a}(b)$  is given by the same formulas as in (1.7), (1.8), except that  $\mathbb{R}^d$  is replaced by  $\Lambda_N$ .

From (2.2)–(2.4) we get

$$(2.5) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{t^{(d-2)/d}} \log P(|W^a(t)| \leq bt) \leq -I_N^{\kappa_a}(b) \quad \text{for all } N > 0.$$

In Section 2.6 we shall show:

PROPOSITION 2.  $\lim_{N \rightarrow \infty} I_N^{\kappa a}(b) = I^{\kappa a}(b)$  for all  $a > 0$  and  $b > 0$ .

Combining this with (2.5) we get the upper bound in Theorem 1.

Our proof of Proposition 1 is based on a new approach for treating large deviations of the Wiener sausage on the torus. This approach uses a conditioning argument, a version of Talagrand's concentration inequality, and the most basic LDP (Large Deviation Principle) of Donsker and Varadhan.

Throughout the rest of this section the Brownian motion lives on  $\Lambda_N$  with  $N$  fixed, and we suppress the indices  $a$  and  $N$  from most expressions. Abbreviate

$$(2.6) \quad V_\tau = |W_N^{a\tau^{-1/(d-2)}}(\tau)|.$$

We shall prove the following:

PROPOSITION 3.  $(V_\tau)_{\tau > 0}$  satisfies the LDP on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(2.7) \quad J_N^{\kappa a}(b) = \inf_{\phi \in \partial\Phi_N^{\kappa a}(b)} \left[ \frac{1}{2} \int_{\Lambda_N} |\nabla \phi|^2(x) dx \right]$$

with

$$(2.8) \quad \partial\Phi_N^{\kappa a}(b) = \left\{ \phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x) dx = 1, \int_{\Lambda_N} \left( 1 - e^{-\kappa a \phi^2(x)} \right) dx = b \right\}.$$

Proposition 3 obviously implies Proposition 1. We shall see in Section 3 that it is also the key to the lower bound in Theorem 1, but this requires a separate argument.

The form of Proposition 3 suggests that some kind of contraction principle is in force. However, it seems to be impossible to approach the problem directly from that angle. Instead, we use an approximation argument consisting of three steps:

- Section 2.2: For  $\varepsilon > 0$ ,

$$(2.9) \quad \mathbb{X}_{\tau, \varepsilon} = \{\beta(i\varepsilon)\}_{1 \leq i \leq \tau/\varepsilon}.$$

(For notational convenience  $\tau/\varepsilon$  is taken to be integral.) We first approximate  $V_\tau$  by  $\mathbb{E}_{\tau, \varepsilon}(V_\tau)$ , where  $\mathbb{E}_{\tau, \varepsilon}$  denotes the *conditional* expectation given  $\mathbb{X}_{\tau, \varepsilon}$ . We prove that the difference between  $V_\tau$  and  $\mathbb{E}_{\tau, \varepsilon}(V_\tau)$  is negligible in the limit as  $\tau \rightarrow \infty$  followed by  $\varepsilon \downarrow 0$ . This is done by application of a concentration inequality of Talagrand.

- Section 2.3: We represent  $\mathbb{E}_{\tau, \varepsilon}(V_\tau)$  as a functional of the empirical measure

$$(2.10) \quad L_{\tau, \varepsilon} = \frac{\varepsilon}{\tau} \sum_{i=1}^{\tau/\varepsilon} \delta_{\beta((i-1)\varepsilon), \beta(i\varepsilon)}.$$

According to Donsker and Varadhan,  $(L_{\tau,\varepsilon})_{\tau>0}$  satisfies an LDP. We need some further approximations to get the dependence of  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$  on  $L_{\tau,\varepsilon}$  in a suitable form, but essentially based on just this LDP we get an LDP for  $(\mathbb{E}_{\tau,\varepsilon}(V_\tau))_{\tau>0}$  via a contraction principle.

- Section 2.4: Finally, we have to perform the limit  $\varepsilon \downarrow 0$ . By our previous result we already know that  $V_\tau$  is well approximated by  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$ . It therefore suffices to have an appropriate approximation for the variational formula in the LDP for  $(\mathbb{E}_{\tau,\varepsilon}(V_\tau))_{\tau>0}$ .

In Section 2.5 the above results are collected to prove Proposition 3.

It will be expedient to use the abbreviation

$$(2.11) \quad T_\tau = \tau^{2/(d-2)}.$$

So the radius of our Wiener sausage on  $\Lambda_N$  is  $a/\sqrt{T_\tau}$ .

**2.2. Approximation of  $V_\tau$  by  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$ .** Recall the definition of  $\mathbb{X}_{\tau,\varepsilon}$  in (2.9). We denote by  $\mathbb{P}_{\tau,\varepsilon}$  and  $\mathbb{E}_{\tau,\varepsilon}$  the conditional probability and expectation given  $\mathbb{X}_{\tau,\varepsilon}$ .

The main result of this section is that  $V_\tau$  is well approximated by  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$  in the following sense:

**PROPOSITION 4.** *For all  $\delta > 0$ ,*

$$(2.12) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|V_\tau - \mathbb{E}_{\tau,\varepsilon}(V_\tau)| \geq \delta) = -\infty.$$

*Proof.* The proof proceeds via a series of estimates.

1. We begin by truncating the excursions. Define

$$(2.13) \quad W_i = \bigcup_{(i-1)\varepsilon \leq s \leq i\varepsilon} B_{a/\sqrt{T_\tau}}(\beta(s)) \quad (1 \leq i \leq \tau/\varepsilon).$$

Then

$$(2.14) \quad V_\tau = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i \right|.$$

For  $K > 0$ , let

$$(2.15) \quad J_{\tau,\varepsilon}^K = \{1 \leq i \leq \tau/\varepsilon: |\beta((i-1)\varepsilon) - \beta(i\varepsilon)| \leq K\sqrt{\varepsilon}\}$$

and define

$$(2.16) \quad V_{\tau,\varepsilon}^K = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i 1\{i \in J_{\tau,\varepsilon}^K\} \right|, \quad \hat{V}_{\tau,\varepsilon}^K = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i 1\{i \notin J_{\tau,\varepsilon}^K\} \right|.$$

Since  $0 \leq V_\tau - V_{\tau,\varepsilon}^K \leq \widehat{V}_{\tau,\varepsilon}^K$ , we have

$$(2.17) \quad |V_\tau - \mathbb{E}_{\tau,\varepsilon}(V_\tau)| \leq |V_{\tau,\varepsilon}^K - \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K)| + \widehat{V}_{\tau,\varepsilon}^K + \mathbb{E}_{\tau,\varepsilon}(\widehat{V}_{\tau,\varepsilon}^K).$$

The claim will follow after we prove the following two results:

(2.18)

$$\lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|V_{\tau,\varepsilon}^K - \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K)| \geq \delta) = -\infty \quad \text{for all } \delta > 0, K \geq K_0(\delta)$$

$$\lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(\widehat{V}_{\tau,\varepsilon}^K \geq \delta) = -\infty \quad \text{for all } \delta > 0, K \geq K_0(\delta).$$

Indeed, the third term on the right-hand side of (2.17) needs no extra consideration, because  $\widehat{V}_{\tau,\varepsilon}^K \leq |\Lambda_N|$  implies that  $\mathbb{E}_{\tau,\varepsilon}(\widehat{V}_{\tau,\varepsilon}^K) \leq \frac{\delta}{2} + |\Lambda_N| \mathbb{P}_{\tau,\varepsilon}(\widehat{V}_{\tau,\varepsilon}^K \geq \frac{\delta}{2})$  and hence

(2.19)

$$P\left(\mathbb{E}_{\tau,\varepsilon}(\widehat{V}_{\tau,\varepsilon}^K) \geq \delta\right) \leq P\left(\mathbb{P}_{\tau,\varepsilon}(\widehat{V}_{\tau,\varepsilon}^K \geq \frac{\delta}{2}) \geq \frac{\delta}{2|\Lambda_N|}\right) \leq \frac{2|\Lambda_N|}{\delta} P\left(\widehat{V}_{\tau,\varepsilon}^K \geq \frac{\delta}{2}\right).$$

2. To prove the second claim in (2.18), we estimate

(2.20)

$$\begin{aligned} P(\widehat{V}_{\tau,\varepsilon}^K \geq \delta) &\leq e^{-\delta\tau/2\varepsilon} E\left(\exp\left[\frac{\tau}{2\varepsilon} \sum_{i=1}^{\tau/\varepsilon} |W_i| 1\{i \notin J_{\tau,\varepsilon}^K\}\right]\right) \\ &= e^{-\delta\tau/2\varepsilon} \left\{ E\left(\exp\left[\frac{\tau}{2\varepsilon} |W_1| 1\{1 \notin J_{\tau,\varepsilon}^K\}\right]\right) \right\}^{\tau/\varepsilon} \\ &= e^{-\delta\tau/2\varepsilon} \left\{ 1 + E\left(1\{1 \notin J_{\tau,\varepsilon}^K\} \left(\exp\left[\frac{\tau}{2\varepsilon} |W_1|\right]\right) - 1\right) \right\}^{\tau/\varepsilon} \\ &\leq e^{-\delta\tau/2\varepsilon} \left\{ 1 + \sqrt{\delta_K C_{\tau,\varepsilon}} \right\}^{\tau/\varepsilon}, \end{aligned}$$

where

$$(2.21) \quad \delta_K = P(|\beta(\varepsilon)| > K\sqrt{\varepsilon})$$

$$C_{\tau,\varepsilon} = E\left(\exp\left[\frac{\tau}{\varepsilon} |W^{a/\sqrt{T_\tau}}(\varepsilon)|\right]\right).$$

It is evident that  $|W^{a/\sqrt{T_\tau}}(\varepsilon)|$  is smaller on the torus than on  $\mathbb{R}^d$ . Therefore we get, after Brownian scaling and using that  $\tau/T_\tau^{d/2} = 1/T_\tau$  by (2.11),

$$(2.22) \quad C_{\tau,\varepsilon} \leq E\left(\exp\left[\frac{1}{\varepsilon T_\tau} |W^a(\varepsilon T_\tau)|\right]\right).$$

It follows from the results in van den Berg and Bolthausen [2] that

$$(2.23) \quad \sup_{T \geq 1} E\left(\exp\left[\frac{1}{T} |W^a(T)|\right]\right) < \infty.$$

Hence the right-hand side of (2.22) is bounded above by some  $C < \infty$  for all  $\tau \geq \tau_0(\varepsilon)$ , and so we find that

$$(2.24) \quad \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(\hat{V}_{\tau,\varepsilon}^K \geq \delta) \leq -\frac{\delta}{2\varepsilon} + \frac{\sqrt{\delta_K C}}{\varepsilon} \quad \text{for all } \varepsilon, K > 0.$$

Since  $\lim_{K \rightarrow \infty} \delta_K = 0$ , there exists a  $K_0(\delta)$  such that  $\sqrt{\delta_K C} \leq \frac{\delta}{4}$  for  $K \geq K_0(\delta)$ . For such  $K$  we now let  $\varepsilon \downarrow 0$  to get the second claim in (2.18).

3. To prove the first claim in (2.18) we argue as follows. Conditionally on  $\mathbb{X}_{\tau,\varepsilon}$ , the  $W_i$  are independent random open subsets of  $\Lambda_N$ . Let  $S$  be the set of open subsets of  $\Lambda_N$ . The mapping  $d: S \times S \mapsto [0, \infty)$  with  $d(A, B) = |A \Delta B|$  defines a pseudometric on  $S$ . We equip  $S$  with the Borel field  $\mathfrak{S}$  generated by this pseudometric. Then  $\mathbb{P}_{\tau,\varepsilon}$  defines a product measure on  $(S, \mathfrak{S})^{\tau/\varepsilon}$ , which we denote by the same symbol  $\mathbb{P}_{\tau,\varepsilon}$ . Define

$$(2.25) \quad V(C) = \left| \bigcup_{i \in J_{\tau,\varepsilon}^K} C_i \right| \quad (C = \{C_i\} \in S^{\tau/\varepsilon})$$

(note that  $\mathbb{X}_{\tau,\varepsilon}$  fixes  $J_{\tau,\varepsilon}^K$ ). Clearly,  $V$  is Lipschitz in the sense that

$$(2.26) \quad |V(C) - V(C')| \leq \sum_{i \in J_{\tau,\varepsilon}^K} |C_i \Delta C'_i| \quad (C, C' \in S^{\tau/\varepsilon}).$$

4. Let us denote by  $m_{\tau,\varepsilon}^K$  the median of the distribution of  $V_{\tau,\varepsilon}^K$  under the conditional law  $\mathbb{P}_{\tau,\varepsilon}$ . Define

$$(2.27) \quad A = \{C \in S^{\tau/\varepsilon}: V(C) \leq m_{\tau,\varepsilon}^K\}.$$

Since the distribution of  $V_{\tau,\varepsilon}^K$  under  $\mathbb{P}_{\tau,\varepsilon}$  has no atoms, we have  $\mathbb{P}_{\tau,\varepsilon}(A) = \frac{1}{2}$ . From Talagrand [25, Th. 2.4.1] (see also Remark 2.1.3) we therefore have

$$(2.28) \quad \mathbb{E}_{\tau,\varepsilon} \left( \exp[\lambda f(A, \{W_i\})] \right) \leq 2 \prod_{i \in J_{\tau,\varepsilon}^K} \mathbb{E}_{\tau,\varepsilon} \left( \cosh[\lambda |W_i \Delta W'_i|] \right),$$

where

$$(2.29) \quad f(A, \{C_i\}) = \inf_{C' \in A} \sum_{i \in J_{\tau,\varepsilon}^K} |C_i \Delta C'_i|$$

and  $\{W'_i\}$  is an independent copy of  $\{W_i\}$ . From the Markov inequality we therefore get

$$(2.30) \quad \mathbb{P}_{\tau,\varepsilon}(f(A, \{W_i\}) \geq \delta) \leq 2 \inf_{\lambda > 0} e^{-\lambda \delta} \prod_{i \in J_{\tau,\varepsilon}^K} \mathbb{E}_{\tau,\varepsilon} \left( \cosh[\lambda |W_i \Delta W'_i|] \right) =: \Xi_{\tau,\varepsilon}^K(\delta).$$

Arguing similarly with  $\hat{A} = \{C \in S^{\tau/\varepsilon}: V(C) \geq m_{\tau,\varepsilon}^K\}$  we get, by (2.26), (2.27),

$$(2.31) \quad \mathbb{P}_{\tau,\varepsilon}(|V_{\tau,\varepsilon}^K - m_{\tau,\varepsilon}^K| \geq \delta) \leq \mathbb{P}_{\tau,\varepsilon}(f(A, \{W_i\}) \geq \delta) + \mathbb{P}_{\tau,\varepsilon}(f(\hat{A}, \{W_i\}) \geq \delta) \leq 2\Xi_{\tau,\varepsilon}^K(\delta).$$

5. Next, since  $V_{\tau,\varepsilon}^K \leq |\Lambda_N|$  we have

$$(2.32) \quad \left| \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K) - m_{\tau,\varepsilon}^K \right| \leq \frac{\delta}{3} + |\Lambda_N| \mathbb{P}_{\tau,\varepsilon} \left( \left| V_{\tau,\varepsilon}^K - m_{\tau,\varepsilon}^K \right| \geq \frac{\delta}{3} \right)$$

and consequently

$$\begin{aligned} (2.33) \quad & \mathbb{P}_{\tau,\varepsilon} \left( \left| V_{\tau,\varepsilon}^K - \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K) \right| \geq \delta \right) \\ & \leq \mathbb{P}_{\tau,\varepsilon} \left( \left| V_{\tau,\varepsilon}^K - m_{\tau,\varepsilon}^K \right| \geq \frac{\delta}{3} \right) + 1 \left\{ \left| \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K) - m_{\tau,\varepsilon}^K \right| \geq \frac{2\delta}{3} \right\} \\ & \leq 2\Xi_{\tau,\varepsilon}^K \left( \frac{\delta}{3} \right) + 1 \left\{ \mathbb{P}_{\tau,\varepsilon} \left( \left| V_{\tau,\varepsilon}^K - m_{\tau,\varepsilon}^K \right| \geq \frac{\delta}{3} \right) \geq \frac{\delta}{3|\Lambda_N|} \right\} \\ & \leq 2\Xi_{\tau,\varepsilon}^K \left( \frac{\delta}{3} \right) + 1 \left\{ 2\Xi_{\tau,\varepsilon}^K \left( \frac{\delta}{3} \right) \geq \frac{\delta}{3|\Lambda_N|} \right\}. \end{aligned}$$

Using this inequality we get, after averaging over  $\mathbb{X}_{\tau,\varepsilon}$ ,

$$(2.34) \quad P(\left| V_{\tau,\varepsilon}^K - \mathbb{E}_{\tau,\varepsilon}(V_{\tau,\varepsilon}^K) \right| \geq \delta) \leq \left( 1 + \frac{3|\Lambda_N|}{\delta} \right) E \left( 2\Xi_{\tau,\varepsilon}^K \left( \frac{\delta}{3} \right) \right).$$

In order to prove the first claim in (2.18), it therefore suffices to show that

$$(2.35) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log E \left( \Xi_{\tau,\varepsilon}^K(\delta) \right) = -\infty \quad \text{for all } \delta > 0, K \geq K_0(\delta).$$

We shall actually prove more, namely that

$$(2.36) \quad \lim_{\varepsilon \downarrow 0} \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} \log \left\| \Xi_{\tau,\varepsilon}^K(\delta) \right\|_{\infty} = -\infty \quad \text{for all } \delta > 0, K \geq K_0(\delta).$$

6. In order to estimate  $\mathbb{E}_{\tau,\varepsilon}(\cosh[\lambda |W_i \Delta W'_i|])$  in (2.30), we pick  $\lambda = c\tau/\varepsilon$  with  $0 < c \leq 1$  and use the fact that  $\cosh(cd) \leq 1 + c^2 \exp(d)$  for  $0 < c \leq 1$  and  $d > 0$ . For  $x \in \Lambda_N$ , we write  $E_{x,\varepsilon}$  to denote expectation under a Brownian bridge of length  $\varepsilon$  between 0 and  $x$ , i.e., a Brownian motion starting at 0 and conditioned to be at  $x$  at time  $\varepsilon$ . It is evident that the volume of the Wiener sausage associated with such a Brownian bridge is smaller on the torus than on  $\mathbb{R}^d$ . Thus we have

$$\begin{aligned} (2.37) \quad & \mathbb{E}_{\tau,\varepsilon} \left( \cosh \left[ c \frac{\tau}{\varepsilon} |W_i \Delta W'_i| \right] \right) \\ & \leq 1 + c^2 \mathbb{E}_{\tau,\varepsilon} \left( \exp \left[ \frac{\tau}{\varepsilon} |W_i \Delta W'_i| \right] \right) \\ & \leq 1 + c^2 \left\{ \mathbb{E}_{\tau,\varepsilon} \left( \exp \left[ \frac{\tau}{\varepsilon} |W_i| \right] \right) \right\}^2 \\ & \leq 1 + c^2 \left\{ \sup_{|x| \leq K} E_{x\sqrt{\varepsilon},\varepsilon} \left( \exp \left[ \frac{\tau}{\varepsilon} |W^{a/\sqrt{T_\tau}}(\varepsilon)| \right] \right) \right\}^2, \end{aligned}$$

where we recall (2.13) and use the fact that  $|\beta((i-1)\varepsilon) - \beta(i\varepsilon)| \leq K\sqrt{\varepsilon}$  for  $i \in J_{\tau,\varepsilon}^K$ . By Brownian scaling we get  $(\tau/T_\tau^{d/2} = 1/T_\tau)$

$$(2.38) \quad E_{x\sqrt{\varepsilon},\varepsilon} \left( \exp \left[ \frac{\tau}{\varepsilon} |W^{a/\sqrt{T_\tau}}(\varepsilon)| \right] \right) \leq E_{x\sqrt{\varepsilon T_\tau},\varepsilon T_\tau} \left( \exp \left[ \frac{1}{\varepsilon T_\tau} |W^a(\varepsilon T_\tau)| \right] \right).$$

7. With the help of Lemma 1 below it follows from (2.38) that there exists a  $C_K < \infty$  such that for all  $\tau \geq \tau_0(\varepsilon)$ ,

$$(2.39) \quad \sup_{|x| \leq K} E_{x\sqrt{\varepsilon},\varepsilon} \left( \exp \left[ \frac{\tau}{\varepsilon} |W^{a/\sqrt{T_\tau}}(\varepsilon)| \right] \right) \leq C_K.$$

Therefore, combining (2.30), (2.37) and (2.39) we get

$$(2.40) \quad \begin{aligned} \Xi_{\tau,\varepsilon}^K(\delta) &\leq 2e^{-c\delta\frac{\tau}{\varepsilon}} \prod_{i \in J_{\tau,\varepsilon}^K} \mathbb{E}_{\tau,\varepsilon} \left( \cosh \left[ c\frac{\tau}{\varepsilon} |W_i \Delta W'_i| \right] \right) \\ &\leq 2e^{-c\delta\frac{\tau}{\varepsilon}} \prod_{i=1}^{\tau/\varepsilon} (1 + c^2 C_K^2) \leq 2e^{(-c\delta + c^2 C_K^2)\frac{\tau}{\varepsilon}}. \end{aligned}$$

Pick  $c = \delta/2C_K^2$  and note that there exists a  $K_0(\delta)$  such that  $0 < c \leq 1$  for  $K \geq K_0(\delta)$ . Let  $\tau \rightarrow \infty$  followed by  $\varepsilon \downarrow 0$ , to get (2.36). The proof of Proposition 4 is now complete.  $\square$

We conclude this section with the following fact:

LEMMA 1. *For every  $K > 0$  there exists a  $C_K < \infty$  such that*

$$(2.41) \quad \sup_{T \geq 2} \sup_{|x| \leq K} E_{x\sqrt{T},T} \left( \exp \left[ \frac{1}{T} |W^a(T)| \right] \right) \leq C_K.$$

*Proof.* We begin by removing the bridge restriction. Write

$$p_t(x,y) = (2\pi t)^{-d/2} \exp[-|x-y|^2/2t]$$

to denote the heat kernel on  $\mathbb{R}^d$  and put  $p_t(x) = p_t(0,x)$ . Write  $E_{y,t;z,2t}$  to denote expectation under a Brownian motion starting at 0 and conditioned to be at  $y$  at time  $t$  and at  $z$  at time  $2t$ . Then we may estimate

$$(2.42) \quad \begin{aligned} E_{z,2t} \left( \exp \left[ \frac{1}{2t} |W^a(2t)| \right] \right) &= \int_{\mathbb{R}^d} dy \frac{p_t(y)p_t(z-y)}{p_{2t}(z)} E_{y,t;z,2t} \left( \exp \left[ \frac{1}{2t} |W^a(2t)| \right] \right) \\ &\leq \frac{1}{p_{2t}(z)} \int_{\mathbb{R}^d} dy p_t(y) E_{y,t} \left( \exp \left[ \frac{1}{2t} |W^a(t)| \right] \right) \\ &\quad \times p_t(z-y) E_{z-y,t} \left( \exp \left[ \frac{1}{2t} |W^a(t)| \right] \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{p_{2t}(z)} \int_{\mathbb{R}^d} dy \left\{ p_t(y) E_{y,t} \left( \exp \left[ \frac{1}{2t} |W^a(t)| \right] \right) \right\}^2 \\
&\leq \frac{p_t(0)}{p_{2t}(z)} \int_{\mathbb{R}^d} dy p_t(y) E_{y,t} \left( \exp \left[ \frac{1}{t} |W^a(t)| \right] \right) \\
&= \frac{p_t(0)}{p_{2t}(z)} E \left( \exp \left[ \frac{1}{t} |W^a(t)| \right] \right).
\end{aligned}$$

Here we use, respectively, the subadditivity of  $t \mapsto |W^a(t)|$ , Cauchy-Schwarz, Jensen and the bound  $p_t(y) \leq p_t(0)$ . Next put  $z = x\sqrt{T}$ ,  $t = T/2$  in (2.42) and use  $\sup_{|x| \leq K} p_{T/2}(0)/p_T(x\sqrt{T}) = 2^{d/2} \exp(K^2/2)$ , to see that the claim follows from (2.23).  $\square$

**2.3. The LDP for  $(\mathbb{E}_{\tau,\varepsilon}(V_\tau))_{\tau>0}$ .** Let  $I_\varepsilon^{(2)}: \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0, \infty]$  be the entropy function

$$(2.43) \quad I_\varepsilon^{(2)}(\mu) = \begin{cases} h(\mu|\mu_1 \otimes \pi_\varepsilon) & \text{if } \mu_1 = \mu_2 \\ \infty & \text{otherwise,} \end{cases}$$

where  $h(\cdot|\cdot)$  denotes relative entropy between measures,  $\mu_1$  and  $\mu_2$  are the two marginals of  $\mu$ , and  $\pi_\varepsilon(x, dy) = p_\varepsilon(y-x)dy$  is the Brownian transition kernel on  $\Lambda_N$ . Furthermore, for  $\eta > 0$  let  $\Phi_\eta: \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0, \infty)$  be the function

$$(2.44) \quad \Phi_\eta(\mu) = \int_{\Lambda_N} dx \left( 1 - \exp \left[ -\eta \kappa_a \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) \mu(dy, dz) \right] \right)$$

with

$$(2.45) \quad \varphi_\varepsilon(y, z) = \frac{\int_0^\varepsilon ds p_s(-y) p_{\varepsilon-s}(z)}{p_\varepsilon(z-y)}.$$

Our main result in this section is the following:

**PROPOSITION 5.**  $(\mathbb{E}_{\tau,\varepsilon}(V_\tau))_{\tau>0}$  satisfies the LDP on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(2.46) \quad J_\varepsilon(b) = \inf \left\{ \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) : \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N), \Phi_{1/\varepsilon}(\mu) = b \right\}.$$

*Proof.* Throughout the proof,  $c_1, c_2, \dots$  are constants that may depend on  $a, \varepsilon, N$  (which are fixed) but not on any of the other variables.

1. First we approximate  $V_\tau$  by cutting out small holes around the points  $\beta(i\varepsilon)$ ,  $1 \leq i \leq \tau/\varepsilon$ . Fix  $K > 0$ , let

$$(2.47) \quad W_i^{K/\sqrt{T_\tau}} = W_i \setminus \left[ B_{K/\sqrt{T_\tau}} \left( \beta((i-1)\varepsilon) \right) \cup B_{K/\sqrt{T_\tau}} \left( \beta(i\varepsilon) \right) \right]$$

and put

$$(2.48) \quad V_\tau^K = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i^{K/\sqrt{T_\tau}} \right|.$$

Clearly, we have cut out at most  $\tau/\varepsilon + 1$  times the volume of a ball of radius  $K/\sqrt{T_\tau}$ , so

$$(2.49) \quad |V_\tau - V_\tau^K| \leq c_1 K^d / T_\tau,$$

which tends to zero as  $\tau \rightarrow \infty$  and therefore is negligible for our purpose. This cutting procedure is convenient as will become clear later on.

2. For  $y, z \in \Lambda_N$ , define

$$(2.50) \quad q_{\tau,\varepsilon}(y, z) = P_{y,z}(\sigma_{a/\sqrt{T_\tau}} \leq \varepsilon),$$

where  $P_{y,z}(\cdot) = P((\beta(t))_{t \in [0, \varepsilon]} \in \cdot \mid \beta(0) = y, \beta(\varepsilon) = z)$  and  $\sigma_{a/\sqrt{T_\tau}}$  is the first entrance time into  $B_{a/\sqrt{T_\tau}} = B_{a/\sqrt{T_\tau}}(0)$ . We can express  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$  in terms of  $q_{\tau,\varepsilon}(y, z)$  and the empirical measure  $L_{\tau,\varepsilon}$  defined in (2.10) as follows:

$$\begin{aligned} \mathbb{E}_{\tau,\varepsilon}(V_\tau^K) &= \int_{\Lambda_N} dx \left( 1 - \mathbb{P}_{\tau,\varepsilon} \left( x \notin \bigcup_{i=1}^{\tau/\varepsilon} W_i^{K/\sqrt{T_\tau}} \right) \right) \\ &= \int_{\Lambda_N} dx \left( 1 - \prod_{i=1}^{\tau/\varepsilon} \left\{ 1 - \mathbb{P}_{\tau,\varepsilon} \left( x \in W_i^{K/\sqrt{T_\tau}} \right) \right\} \right) \\ &= \int_{\Lambda_N} dx \left( 1 - \exp \left[ \frac{\tau}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \log \left( 1 - q_{\tau,\varepsilon}^{K/\sqrt{T_\tau}}(y-x, z-x) \right) L_{\tau,\varepsilon}(dy, dz) \right] \right), \end{aligned} \quad (2.51)$$

where for  $\rho > 0$  we define  $q_{\tau,\varepsilon}^\rho(y, z) = q_{\tau,\varepsilon}(y, z)$  if  $y, z \notin B_\rho$  and zero otherwise.

3. We want to expand the logarithm and do an approximation. For this we need the following facts about Brownian motion on  $\Lambda_N$ , which come as an intermezzo. Recall that  $\kappa_a$  is the Newtonian capacity of the ball with radius  $a$ .

LEMMA 2. (a)  $\lim_{K \rightarrow \infty} \limsup_{\tau \rightarrow \infty} \sup_{y, z \notin B_{K/\sqrt{T_\tau}}} q_{\tau,\varepsilon}(y, z) = 0$ .

(b)  $\lim_{\tau \rightarrow \infty} \sup_{y, z \notin B_\rho} |\tau q_{\tau,\varepsilon}(y, z) - \kappa_a \varphi_\varepsilon(y, z)| = 0$  for all  $0 < \rho < N/4$ .

*Proof.* (a) Throughout the proof  $\varepsilon, N$  are fixed.

i. We begin by removing both the bridge restriction and the torus restriction. For  $y, z \in \Lambda_N$  and  $0 < b < N/2$ , let

$$(2.52) \quad q_b(y, z) = P_{y,z}(\sigma_b \leq \varepsilon),$$

where  $\sigma$  is the first entrance time into  $B_b$ . There exists a constant  $c_2$  such that  $\frac{dP_{y,z}}{dP_y} \leq c_2$ , where  $P_y(\cdot) = P((\beta(t))_{t \in [0, \varepsilon/2]} \in \cdot \mid \beta(0) = y)$ . Hence

$$(2.53) \quad \sup_{y, z \notin B_{b'}} q_b(y, z) \leq 2c_2 \sup_{y \notin B_{b'}} P_y(\sigma_b \leq \varepsilon/2) \quad \text{for all } 0 < b < b' < N/4.$$

Let  $\widehat{\sigma}_{N/2}$  be the first entrance time into  $B_{N/2}^c = \Lambda_N \setminus B_{N/2}$ . Then for any  $y \notin B_{b'}$  we may decompose

$$(2.54) \quad P_y(\sigma_b \leq \varepsilon/2) = P_y(\sigma_b \leq \varepsilon/2, \sigma_b < \widehat{\sigma}_{N/2}) + P_y(\sigma_b \leq \varepsilon/2, \sigma_b \geq \widehat{\sigma}_{N/2}).$$

To estimate the second term on the right-hand side, we note that on its way from  $\partial B_{N/2}$  to  $\partial B_b$  the Brownian motion must first cross  $\partial B_{N/4}$  and then cross  $\partial B_{b'}$ . Hence for any  $y \notin B_{b'}$ ,

$$(2.55) \quad P_y(\sigma_b \leq \varepsilon/2, \sigma_b \geq \widehat{\sigma}_{N/2}) \leq c_3 \sup_{x \in \partial B_{b'}} P_x(\sigma_b \leq \varepsilon/2)$$

with  $c_3 = \sup_{x \in \partial B_{N/2}} P_x(\sigma_{N/4} \leq \varepsilon/2)$ . Evidently,  $c_3 < 1$  and so we deduce from (2.54) and (2.55) that

$$(2.56) \quad \sup_{y \notin B_{b'}} P_y(\sigma_b \leq \varepsilon/2) \leq \frac{1}{1 - c_3} \sup_{y \notin B_{b'}} P_y(\sigma_b \leq \varepsilon/2, \sigma_b < \widehat{\sigma}_{N/2}).$$

As long as the Brownian motion does not hit  $B_{N/2}^c$  it behaves like a Brownian motion on  $\mathbb{R}^d$ . Therefore

$$(2.57) \quad P_y(\sigma_b \leq \varepsilon/2, \sigma_b < \widehat{\sigma}_{N/2}) \leq P_y^\infty(\sigma_b \leq \varepsilon/2),$$

where the upper index  $\infty$  refers to removal of the torus restriction. Combining (2.53), (2.56) and (2.57) we arrive at

$$(2.58) \quad \sup_{y, z \notin B_{b'}} q_b(y, z) \leq \frac{2c_2}{1 - c_3} \sup_{y \notin B_{b'}} P_y^\infty(\sigma_b \leq \varepsilon/2) \quad \text{for all } 0 < b < b' < N/4.$$

ii. Since

$$(2.59) \quad P_y^\infty(\sigma_b \leq \varepsilon/2) \leq P_y^\infty(\sigma_b < \infty) = \left( \frac{b}{|y|} \right)^{d-2},$$

we obtain from (2.58) that

$$(2.60) \quad \sup_{y, z \notin B_{b'}} q_b(y, z) \leq \frac{2c_2}{1 - c_3} \left( \frac{b}{b'} \right)^{d-2}.$$

Now put  $b = a/\sqrt{T_\tau}$ ,  $b' = K/\sqrt{T_\tau}$  and take the limit  $K \rightarrow \infty$ , to get the claim in Part (a).

(b) Throughout the proof  $\varepsilon, N$  are again fixed.

i. We shall prove that

$$(2.61) \quad \lim_{b \downarrow 0} \sup_{y, z \notin B_\rho} \left| \frac{q_b(y, z)}{\kappa_b} - \varphi_\varepsilon(y, z) \right| = 0 \quad \text{for all } \rho > 0.$$

Put  $b = a/\sqrt{T_\tau}$  in (2.61) (recall (2.50)) and use the fact that  $\kappa_{a/\sqrt{T_\tau}} = \kappa_a/\tau$  (recall (2.11)) to get the claim in Part (b).

ii. Let  $0 < \delta < \varepsilon/2$ . Define

$$(2.62) \quad q_b^\delta(y, z) = P_{y,z}(\sigma_b \in [\delta, \varepsilon - \delta]).$$

Then, by the argument in Step i of Part (a),

$$\begin{aligned} (2.63) \quad \sup_{y,z \notin B_\rho} |q_b(y, z) - q_b^\delta(y, z)| &\leq \sup_{y,z \notin B_\rho} P_{y,z}(\exists s \in [0, \delta] \cap [\varepsilon - \delta, \varepsilon]: \beta(s) \in B_b) \\ &\leq \frac{2c_2}{1-c_3} \sup_{y \notin B_\rho} P_y^\infty(\sigma_b \leq \delta). \end{aligned}$$

The supremum on the right-hand side is taken at any  $y_0 \in \partial B_\rho$ . We may now invoke a result by Le Gall [16], which says that

$$(2.64) \quad \lim_{b \downarrow 0} \frac{1}{\kappa_b} P_y^\infty(\sigma_b \leq t) = \int_0^t p_s(-y) ds \quad \text{for all } y \in \mathbb{R}^d, t \geq 0.$$

This gives us

$$(2.65) \quad \lim_{b \downarrow 0} \frac{1}{\kappa_b} \sup_{y,z \notin B_\rho} |q_b(y, z) - q_b^\delta(y, z)| \leq \frac{2c_2}{1-c_3} \int_0^\delta p_s(-y_0) ds.$$

We thus see that to prove (2.61) it suffices to show that

$$(2.66) \quad \lim_{\delta \downarrow 0} \lim_{b \downarrow 0} \sup_{y,z \notin B_\rho} \left| \frac{q_b^\delta(y, z)}{\kappa_b} - \varphi_\varepsilon(y, z) \right| = 0 \quad \text{for all } \rho > 0.$$

iii. To analyze  $q_b^\delta(y, z)$ , we make a first entrance decomposition on  $\partial B_b$ :

$$(2.67) \quad q_b^\delta(y, z) = \frac{1}{p_\varepsilon(z-y)} \int_\delta^{\varepsilon-\delta} \int_{\partial B_b} P_y(\sigma_b \in ds, \beta(\sigma_b) \in dx) p_{\varepsilon-s}(z-x).$$

Next we note that  $p_{\varepsilon-s}(z-x) = [1 + o_\delta(1)]p_{\varepsilon-s}(z)$  uniformly in  $z \notin B_\rho$ ,  $x \in \partial B_b$ ,  $s \in [\delta, \varepsilon - \delta]$ , where the  $o_\delta(1)$  refers to  $b \downarrow 0$  for fixed  $\delta$ . Inserting this approximation into (2.67), we get

$$(2.68) \quad q_b^\delta(y, z) = \frac{1 + o_\delta(1)}{p_\varepsilon(z-y)} \int_\delta^{\varepsilon-\delta} P_y(\sigma_b \in ds) p_{\varepsilon-s}(z).$$

For the full integral we have

$$(2.69) \quad \int_0^\varepsilon P_y(\sigma_b \in ds) p_{\varepsilon-s}(z) = \int_0^\varepsilon ds P_y(\sigma_b \leq s) \left[ \frac{\partial}{\partial s} p_{\varepsilon-s}(z) \right]$$

and so using (2.64) we get, by dominated convergence,

$$\begin{aligned} (2.70) \quad \lim_{b \downarrow 0} \frac{1}{\kappa_b} \int_0^\varepsilon P_y(\sigma_b \in ds) p_{\varepsilon-s}(z) &= \int_0^\varepsilon ds \int_0^s du p_u(-y) \left[ \frac{\partial}{\partial s} p_{\varepsilon-s}(z) \right] \\ &= \int_0^\varepsilon ds p_s(-y) p_{\varepsilon-s}(z). \end{aligned}$$

The limit is in fact uniform in  $y, z \notin B_\rho$ , because  $\Lambda_N$  is a compact set. Therefore, recalling (2.68) and the definition of  $\varphi_\varepsilon(y, z)$  in (2.45), we see that to prove (2.66) it suffices to show that uniformly in  $y, z \notin B_\rho$ ,

$$(2.71) \quad \lim_{\delta \downarrow 0} \lim_{b \downarrow 0} \frac{1}{\kappa_b} \int_0^\delta P_y(\sigma_b \in ds) p_{\varepsilon-s}(z) = 0,$$

and similarly for the integral over  $[\varepsilon - \delta, \varepsilon]$ . However, the second factor is bounded uniformly in  $z \notin B_\rho$  and  $s \in [0, \delta]$ , and so we are left with  $\frac{1}{\kappa_b} P_y(\sigma_b \leq \delta)$ . Since  $P_y(\sigma_b \leq \delta) \leq \frac{1}{1-c_3} P_y^\infty(\sigma_b \leq \delta)$  for all  $0 < \rho < N/4$ , by the argument in Step i of Part (a), we indeed get (2.71) via another application of (2.64).  $\square$

4. We pick up the line of proof left off at the end of Step 2. From Lemma 2(a) it follows that there exists  $\delta_K > 0$ , satisfying  $\lim_{K \rightarrow \infty} \delta_K = 0$ , such that

$$(2.72) \quad -(1 + \delta_K) q_{\tau, \varepsilon}^{K/\sqrt{T_\tau}} \leq \log(1 - q_{\tau, \varepsilon}^{K/\sqrt{T_\tau}}) \leq -q_{\tau, \varepsilon}^{K/\sqrt{T_\tau}}.$$

We are therefore naturally led to an investigation of the functions

$$\Phi_{\tau, \eta, \rho} : \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0, \infty)$$

defined by

$$(2.73) \quad \Phi_{\tau, \eta, \rho}(\mu) = \int_{\Lambda_N} dx \left( 1 - \exp \left[ -\eta \tau \int_{\Lambda_N \times \Lambda_N} q_{\tau, \varepsilon}^\rho(y-x, z-x) \mu(dy, dz) \right] \right),$$

for which (2.51) and (2.72) give us the following sandwich:

$$(2.74) \quad \Phi_{\tau, 1/\varepsilon, K/\sqrt{T_\tau}}(L_{\tau, \varepsilon}) \leq \mathbb{E}_{\tau, \varepsilon}(V_\tau^K) \leq \Phi_{\tau, (1+\delta_K)/\varepsilon, K/\sqrt{T_\tau}}(L_{\tau, \varepsilon}).$$

The functions  $\Phi_{\tau, \eta, \rho}$  have nice continuity properties:

LEMMA 3. *There exist constants  $c_4, c_5$  such that:*

- (a)  $|\Phi_{\tau, \eta, \rho}(\mu) - \Phi_{\tau, \eta, \rho'}(\mu)| \leq c_4 \eta |\sqrt{\rho} - \sqrt{\rho'}|$  for all  $\eta, \mu$  and  $\tau \geq \tau_0(\rho, \rho')$ .
- (b)  $|\Phi_{\tau, \eta, \rho}(\mu) - \Phi_{\tau, \eta', \rho}(\mu)| \leq c_5 |\eta - \eta'|$  for all  $\rho, \mu$  and  $\tau \geq \tau_0(\rho)$ .

*Proof.* The proof uses Lemma 2(b).

(a) Write

$$(2.75) \quad \begin{aligned} & |\Phi_{\tau, \eta, \rho}(\mu) - \Phi_{\tau, \eta, \rho'}(\mu)| \\ & \leq \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) \left| \tau q_{\tau, \varepsilon}^\rho(y-x, z-x) - \tau q_{\tau, \varepsilon}^{\rho'}(y-x, z-x) \right| \\ & = \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) \left[ \left| \kappa_a \varphi_\varepsilon^\rho(y-x, z-x) \right. \right. \\ & \quad \left. \left. - \kappa_a \varphi_\varepsilon^{\rho'}(y-x, z-x) \right| + o_{\rho, \rho'}(1) \right] \\ & = \eta \left[ \left| \sqrt{\rho} - \sqrt{\rho'} \right| + |\Lambda_N| o_{\rho, \rho'}(1) \right]. \end{aligned}$$

Here,  $o_{\rho,\rho'}(1)$  means an error tending to zero as  $\tau \rightarrow \infty$  depending on  $\rho, \rho'$ , and in the last equality we use the fact that  $\int_{\Lambda_N} dx \varphi_\varepsilon(y - x, z - x) = \varepsilon$  for all  $y, z$ .

(b) Write

$$\begin{aligned}
 (2.76) \quad & |\Phi_{\tau,\eta,\rho}(\mu) - \Phi_{\tau,\eta',\rho}(\mu)| \\
 & \leq |\eta - \eta'| \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) \tau q_{\tau,\varepsilon}^\rho(y - x, z - x) \\
 & = |\eta - \eta'| \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) [\kappa_a \varphi_\varepsilon^\rho(y - x, z - x) + o_\rho(1)] \\
 & \leq |\eta - \eta'| [\kappa_a \varepsilon + |\Lambda_N| o_\rho(1)],
 \end{aligned}$$

where in the last inequality we drop the superscript  $\rho$  to be able to perform the  $x$ -integration.  $\square$

5. With the help of (2.49), (2.74) and Lemma 3(a), (b), we get

$$\begin{aligned}
 (2.77) \quad & \mathbb{E}_{\tau,\varepsilon}(V_\tau) \leq \mathbb{E}_{\tau,\varepsilon}(V_\tau^K) + c_1 K^d / T_\tau \\
 & \leq \Phi_{\tau,(1+\delta_K)/\varepsilon, K/\sqrt{T_\tau}}(L_{\tau,\varepsilon}) + c_1 K^d / T_\tau \\
 & \leq \Phi_{\tau,1/\varepsilon,\rho}(L_{\tau,\varepsilon}) + c_1 K^d / T_\tau + c_4 \left[ \sqrt{K/\sqrt{T_\tau}} + \sqrt{\rho} \right] / \varepsilon + c_5 \delta_K / \varepsilon,
 \end{aligned}$$

and also a similar lower bound.

6. Next we approximate  $\Phi_{\tau,1/\varepsilon,\rho}(L_{\tau,\varepsilon})$  by  $\Phi_{\infty,1/\varepsilon,\rho}(L_{\tau,\varepsilon})$  defined as

$$(2.78) \quad \Phi_{\infty,\eta,\rho}(\mu) = \int_{\Lambda_N} dx \left( 1 - \exp \left[ -\eta \kappa_a \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon^\rho(y - x, z - x) \mu(dy, dz) \right] \right),$$

where for  $\rho > 0$  we define  $\varphi_\varepsilon^\rho(y, z) = \varphi_\varepsilon(y, z)$  if  $y, z \notin B_\rho$  and zero otherwise. For that we need the following:

LEMMA 4. *There exist constants  $c_6, c_7 > 0$  such that:*

- (a)  $|\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{\tau,\eta,\rho}(\mu)| \leq c_6 \eta \delta_{\rho,\tau}$  for all  $\mu$  with  $\lim_{\tau \rightarrow \infty} \delta_{\rho,\tau} = 0$  for any  $\rho > 0$ .
- (b)  $|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Phi_{\infty,1/\varepsilon,0}(\mu')| \leq c_7 \|\mu - \mu'\|_{tv}$ , where  $\|\cdot\|_{tv}$  denotes the total variation norm.

*Proof.* The proof again uses Lemma 2(b).

(a) Write

(2.79)

$$\begin{aligned} & |\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{\tau,\eta,\rho}(\mu)| \\ & \leq \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) [\tau q_{\tau,\varepsilon}^\rho(y-x, z-x) - \kappa_a \varphi_\varepsilon(y-x, z-x)] \\ & = \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) o_\rho(1) = \eta |\Lambda_N| o_\rho(1). \end{aligned}$$

(b) Write

$$\begin{aligned} (2.80) \quad & \left| \Phi_{\infty,1/\varepsilon,0}(\mu) - \Phi_{\infty,1/\varepsilon,0}(\mu') \right| \\ & \leq \frac{\kappa_a}{\varepsilon} \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} |\mu - \mu'| (dy, dz) \varphi_\varepsilon(y-x, z-x) \\ & = \kappa_a \int_{\Lambda_N \times \Lambda_N} |\mu - \mu'| (dy, dz) = \kappa_a \|\mu - \mu'\|_{tv}. \end{aligned} \quad \square$$

7. Using (2.77), the similar lower bound and Lemma 4(a) with  $\eta = 1/\varepsilon$ , we now have that for any  $K$  and  $\rho$ ,

$$\begin{aligned} (2.81) \quad & \left\| \mathbb{E}_{\tau,\varepsilon}(V_\tau) - \Phi_{\infty,1/\varepsilon,0}(L_{\tau,\varepsilon}) \right\|_\infty \\ & \leq c_1 K^d / T_\tau + c_4 \left[ \sqrt{K/\sqrt{T_\tau}} + \sqrt{\rho} \right] / \varepsilon + c_5 \delta_K / \varepsilon + c_6 \delta_{\rho,\tau} / \varepsilon. \end{aligned}$$

Letting  $\tau \rightarrow \infty$ , followed by  $K \rightarrow \infty$  and  $\rho \downarrow 0$ , we thus arrive at

$$(2.82) \quad \lim_{\tau \rightarrow \infty} \left\| \mathbb{E}_{\tau,\varepsilon}(V_\tau) - \Phi_{\infty,1/\varepsilon,0}(L_{\tau,\varepsilon}) \right\|_\infty = 0 \quad \text{for all } \varepsilon > 0.$$

8. The desired LDP for fixed  $\varepsilon$  can now be derived as follows. First, note that  $\Phi_{\infty,1/\varepsilon,0}$  is continuous by Lemma 4(b) (even in the total variation topology). Next, note from (2.78) that  $\Phi_{\infty,1/\varepsilon,0} = \Phi_{1/\varepsilon}$ , the function defined in (2.44). Therefore we can use one of the standard results of Donsker and Varadhan [13](III) (see also Bolthausen [5]), namely, that  $(L_{\tau,\varepsilon})_{\tau>0}$  satisfies the LDP on  $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$  with rate  $\tau$  and with rate function  $\frac{1}{\varepsilon} I_\varepsilon^{(2)}$  defined in (2.43). From the contraction principle and (2.82) we now get the claim in Proposition 5.  $\square$

2.4. *The limit  $\varepsilon \downarrow 0$  for the LDP.* We already know from Section 2.2 that the quantity of interest, namely  $V_\tau$ , is for small  $\varepsilon$  well approximated by  $\mathbb{E}_{\tau,\varepsilon}(V_\tau)$ , for which we have the LDP in Proposition 5. The main step to prove the LDP for  $V_\tau$  itself is therefore to derive an appropriate limit result for the rate function in (2.46). This needs some preparations.

1. We denote by  $I: \mathcal{M}_1^+(\Lambda_N) \mapsto [0, \infty]$  the standard large deviation rate function for the empirical distribution of the Brownian motion:

$$(2.83) \quad \begin{aligned} I(\nu) &= \frac{1}{2} \int_{\Lambda_N} |\nabla \phi|^2(x) dx && \text{if } \frac{d\nu}{dx} = \phi^2 \text{ with } \phi \in H^1(\Lambda_N) \\ &= \infty && \text{otherwise.} \end{aligned}$$

We further denote by  $I_\varepsilon: \mathcal{M}_1^+(\Lambda_N) \mapsto [0, \infty]$  the following projection of  $I_\varepsilon^{(2)}$  (recall (2.43)) onto  $\mathcal{M}_1^+(\Lambda_N)$ :

$$(2.84) \quad I_\varepsilon(\nu) = \inf \left\{ I_\varepsilon^{(2)}(\mu): \mu_1 = \nu \right\}.$$

We begin by collecting some basic facts about these entropies, all of which have been proved by Donsker and Varadhan [13] or are simple consequences of their results:

LEMMA 5. *Let  $(\pi_t)_{t \geq 0}$  denote the semigroup of the Brownian motion. Then for all  $\nu, \mu$ :*

- (a)  $I_t(\nu) = -\inf_{u \in \mathcal{D}^+} \int \log \frac{\pi_t u}{u} d\nu$ , where  $\mathcal{D}^+$  is the set of positive measurable functions bounded away from 0 and  $\infty$ .
- (b)  $t \mapsto I_t(\nu)/t$  is nonincreasing with  $\lim_{t \downarrow 0} I_t(\nu)/t = I(\nu)$ .
- (c)  $\|\nu - \nu \pi_s\|_{tv} \leq 8\sqrt{I_s(\nu)}$  for  $s > 0$ .
- (d)  $I_s(\nu \pi_t) \leq I_s(\nu)$  for  $s, t > 0$ .
- (e)  $\|\mu - \mu_1 \otimes \pi_s\|_{tv} \leq 8\sqrt{I_s^{(2)}(\mu)}$  for  $s > 0$ .

*Proof.* (a) This is [13, (III), Th. 2.1], combined with [13, (I), Lemma 2.1].

(b) Fix  $s, t > 0$ . For every  $u \in \mathcal{D}^+$ ,

$$(2.85) \quad \int \log \frac{\pi_{s+t} u}{u} d\nu = \int \log \frac{\pi_s(\pi_t u)}{\pi_t u} d\nu + \int \log \frac{\pi_t u}{u} d\nu \geq -I_s(\nu) - I_t(\nu).$$

Taking the infimum over  $u$  and using (a), we get  $-I_{s+t}(\nu) \geq -I_s(\nu) - I_t(\nu)$ . Hence  $t \mapsto I_t(\nu)/t$  is nonincreasing. The fact that  $\lim_{t \downarrow 0} I_t(\nu)/t = I(\nu)$  is [13, (I), Lemma 3.1].

- (c) This is [13, (I), Lemma 4.1].
- (d) This follows from the convexity of  $\nu \mapsto I_s(\nu)$  for  $s > 0$ .
- (e) Let  $P^\mu(x, dy)$  be any transition kernel on  $\Lambda_N$  such that  $\mu = \mu_1 \otimes P^\mu$ . Then

$$(2.86) \quad \|\mu - \mu_1 \otimes \pi_s\|_{tv} \leq \int \mu_1(dx) \|P^\mu(x, \cdot) - \pi_s(x, \cdot)\|_{tv}.$$

By Csiszár [11, Th. 4.1], we have (recall that  $h(\cdot|\cdot)$  denotes relative entropy)

$$(2.87) \quad \|P^\mu(x, \cdot) - \pi_s(x, \cdot)\|_{tv} \leq 8\sqrt{h(P^\mu(x, \cdot) | \pi_s(x, \cdot))}.$$

Therefore

$$\begin{aligned} (2.88) \quad \|\mu - \mu_1 \otimes \pi_s\|_{tv} &\leq 8 \int \mu_1(dx) \sqrt{h(P^\mu(x, \cdot) | \pi_s(x, \cdot))} \\ &\leq 8 \sqrt{\int \mu_1(dx) h(P^\mu(x, \cdot) | \pi_s(x, \cdot))} = 8\sqrt{I_s^{(2)}(\mu)}, \end{aligned}$$

where the last equality uses (2.43).  $\square$

2. To take advantage of the link provided by Lemma 5(b), we shall need an approximation of the functions  $\Phi_{1/\varepsilon}: \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0, \infty)$ , appearing in Proposition 5, by the simpler functions  $\Psi_{1/\varepsilon}: \mathcal{M}_1^+(\Lambda_N) \mapsto [0, \infty)$  defined by

$$(2.89) \quad \Psi_{1/\varepsilon}(\nu) = \int_{\Lambda_N} dx \left( 1 - \exp \left[ -\frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \int_{\Lambda_N} p_s(x-y) \nu(dy) \right] \right).$$

LEMMA 6. *For any  $K > 0$ ,*

$$(2.90) \quad \lim_{\varepsilon \downarrow 0} \sup_{\mu: \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \leq K} |\Phi_{1/\varepsilon}(\mu) - \Psi_{1/\varepsilon}(\mu)| = 0.$$

*Proof.* As is obvious by comparing (2.44), (2.45) with (2.89), we have  $\Psi_{1/\varepsilon}(\mu_1) = \Phi_{1/\varepsilon}(\mu_1 \otimes \pi_\varepsilon)$ . Therefore

$$\begin{aligned} (2.91) \quad &|\Phi_{1/\varepsilon}(\mu) - \Psi_{1/\varepsilon}(\mu_1)| \\ &= |\Phi_{1/\varepsilon}(\mu) - \Phi_{1/\varepsilon}(\mu_1 \otimes \pi_\varepsilon)| \\ &\leq \frac{\kappa_a}{\varepsilon} \left| \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) [\mu(dy, dz) - (\mu_1 \otimes \pi_\varepsilon)(dy, dz)] \right| \\ &\leq \frac{\kappa_a}{\varepsilon} \int_{\Lambda_N \times \Lambda_N} \left\{ \int_{\Lambda_N} dx \varphi_\varepsilon(y-x, z-x) \right\} |\mu - \mu_1 \otimes \pi_\varepsilon|(dy, dz) \\ &= \kappa_a \|\mu - \mu_1 \otimes \pi_\varepsilon\|_{tv}, \end{aligned}$$

where in the last equality we again use the fact that the integral between braces equals  $\varepsilon$  for all  $y, z$  by (2.45). The claim now follows from Lemma 5(e).  $\square$

3. Next, we define the function  $\Gamma: L_1^+(\Lambda_N) \mapsto [0, \infty)$  by

$$(2.92) \quad \Gamma(f) = \int_{\Lambda_N} dx \left( 1 - e^{-\kappa_a f(x)} \right).$$

LEMMA 7. *For any  $K > 0$ ,*

$$(2.93) \quad \lim_{\varepsilon \downarrow 0} \sup_{\nu: \frac{1}{\varepsilon} I_\varepsilon(\nu) \leq K} \left| \Gamma\left(\frac{d\nu}{dx}\right) - \Psi_{1/\varepsilon}(\nu) \right| = 0.$$

(Note from (2.43) and (2.84) that if  $I_\varepsilon(\nu) < \infty$ , then  $d\nu \ll dx$  because  $\nu \otimes \pi_\varepsilon \ll dx \otimes dy$ .)

*Proof.* Using (2.89) and (2.92), we write

$$\begin{aligned} (2.94) \quad & \left| \Gamma\left(\frac{d\nu}{dx}\right) - \Psi_{1/\varepsilon}(\nu) \right| \\ & \leq \int_{\Lambda_N} dx \left| \exp \left[ -\frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \int_{\Lambda_N} p_s(x-y) \nu(dy) \right] \right. \\ & \quad \left. - \exp \left[ -\frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \frac{d\nu}{dx}(x) \right] \right| \\ & \leq \int_{\Lambda_N} dx \frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \left| \frac{\nu \pi_s}{dx}(x) - \frac{d\nu}{dx}(x) \right| = \frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \|\nu \pi_s - \nu\|_{tv}. \end{aligned}$$

Now, for  $0 \leq s \leq \varepsilon$  we have, by Lemma 5(c),

$$(2.95) \quad \|\nu \pi_s - \nu\|_{tv} \leq \|\nu \pi_s - \nu \pi_{s+\varepsilon}\|_{tv} + \|\nu \pi_{s+\varepsilon} - \nu\|_{tv} \leq 8\sqrt{I_\varepsilon(\nu \pi_s)} + 8\sqrt{I_{\varepsilon+s}(\nu)}.$$

Moreover,  $I_\varepsilon(\nu \pi_s) \leq I_\varepsilon(\nu)$  by Lemma 5(d) and

$$I_{\varepsilon+s}(\nu) \leq 2\varepsilon I_{\varepsilon+s}(\nu)/(\varepsilon+s) \leq 2\varepsilon I_\varepsilon(\nu)/\varepsilon = 2I_\varepsilon(\nu)$$

by Lemma 5(b). Thus we get  $\|\nu \pi_s - \nu\|_{tv} \leq 8(1 + \sqrt{2})\sqrt{I_\varepsilon(\nu)}$ . From this the claim follows.  $\square$

We now have all the ingredients to perform the proof of Proposition 3 in Section 2.1.

2.5. *Proof of Proposition 3.* For any  $f: \mathbb{R}_+ \mapsto \mathbb{R}$  bounded and continuous:

$$\begin{aligned} (2.96) \quad & \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log E(\exp[\tau f(V_\tau)]) = \lim_{\varepsilon \downarrow 0} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log E(\exp[\tau f(\mathbb{E}_{\tau, \varepsilon}(V_\tau))]) \\ & = \limsup_{\varepsilon \downarrow 0} \mu \left\{ f(\Phi_{1/\varepsilon}(\mu)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \\ & = \lim_{K \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \sup_{\mu: \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \leq K} \left\{ f(\Phi_{1/\varepsilon}(\mu)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \\ & = \lim_{K \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \sup_{\mu: \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \leq K} \left\{ f(\Psi_{1/\varepsilon}(\mu_1)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \\ & = \lim_{K \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \sup_{\nu: \frac{1}{\varepsilon} I_\varepsilon(\nu) \leq K} \left\{ f(\Psi_{1/\varepsilon}(\nu)) - \frac{1}{\varepsilon} I_\varepsilon(\nu) \right\} \end{aligned}$$

$$\begin{aligned}
&= \lim_{K \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \sup_{\nu: \frac{1}{\varepsilon} I_\varepsilon(\nu) \leq K} \left\{ f \left( \Gamma \left( \frac{d\nu}{dx} \right) \right) - \frac{1}{\varepsilon} I_\varepsilon(\nu) \right\} \\
&= \sup_{\nu} \left\{ f \left( \Gamma \left( \frac{d\nu}{dx} \right) \right) - I(\nu) \right\} \\
&= \sup_{\phi \in H^1(\Lambda_N): \|\phi\|_2^2 = 1} \left\{ f(\phi^2) - \frac{1}{2} \|\nabla \phi\|_2^2 \right\}.
\end{aligned}$$

Here we use, respectively, Proposition 4, Proposition 5, Lemma 6, equation (2.84), Lemma 7, Lemma 5(b) and equation (2.83). Recalling (2.92), we see that the claim now follows from the inverse of Varadhan's lemma proved in Bryc [10].  $\square$

**2.6. Proof of Proposition 2.** Throughout this section,  $a > 0$  and  $b > 0$  are fixed. For ease of notation we introduce the following abbreviations:

$$(2.97) \quad A(\phi) = \int_{\mathbb{R}^d} \phi^2(x) dx, \quad B(\phi) = \int_{\mathbb{R}^d} (1 - e^{-\kappa_a \phi^2(x)}) dx, \quad C(\phi) = \int_{\mathbb{R}^d} |\nabla \phi|^2(x) dx$$

for  $\phi \in H^1(\mathbb{R}^d)$ , and their counterparts  $A_N(\phi_N)$ ,  $B_N(\phi_N)$ ,  $C_N(\phi_N)$  for  $\phi_N \in H^1(\Lambda_N)$  with  $\Lambda_N = [-\frac{N}{2}, \frac{N}{2}]^d$ , the  $N$ -torus with periodic boundary conditions.

1. ' $I_N^{\kappa_a}(b) \leq I^{\kappa_a}(b)$  for all  $N > 0$ '.

For  $\phi \in H^1(\mathbb{R}^d)$ , let  $\sigma_N \phi \in H^1(\Lambda_N)$  be defined by

$$(2.98) \quad (\sigma_N \phi)^2(x) = \begin{cases} \sum_{k \in \mathbb{Z}^d} \phi^2(x + kN) & (x \in \Lambda_N) \\ 0 & (x \notin \Lambda_N). \end{cases}$$

Then

$$(2.99) \quad A_N(\sigma_N \phi) = A(\phi), \quad B_N(\sigma_N \phi) \leq B(\phi), \quad C_N(\sigma_N \phi) \leq C(\phi),$$

where the second and third statements hold because  $1 - e^{-(f+g)} \leq (1 - e^{-f}) + (1 - e^{-g})$ , respectively,  $(\nabla \sqrt{f^2 + g^2})^2 \leq (\nabla f)^2 + (\nabla g)^2$  for arbitrary functions  $f, g \geq 0$ . Hence

$$\begin{aligned}
(2.100) \quad I_N^{\kappa_a}(b) &= \inf \{ C_N(\phi_N): \phi_N \in H^1(\Lambda_N), A_N(\phi_N) = 1, B_N(\phi_N) \leq b \} \\
&= \inf \{ C_N(\sigma_N \phi): \phi \in H^1(\mathbb{R}^d), A_N(\sigma_N \phi) = 1, B_N(\sigma_N \phi) \leq b \} \\
&\leq \inf \{ C(\phi): \phi \in H^1(\mathbb{R}^d), A(\phi) = 1, B(\phi) \leq b \} = I^{\kappa_a}(b).
\end{aligned}$$

2. ' $\liminf_{N \rightarrow \infty} I_N^{\kappa_a}(b) \geq I^{\kappa_a}(b+)$ '.

For every  $\varepsilon > 0$  there exists a  $\phi_N \in H^1(\Lambda_N)$  such that

$$(2.101) \quad A_N(\phi_N) = 1, \quad B_N(\phi_N) \leq b, \quad C_N(\phi_N) \leq I_N^{\kappa_a}(b) + \varepsilon,$$

i.e.,  $\phi_N$  is an  $\varepsilon$ -minimiser. By shifting  $\Lambda_N$  around, we see that there must exist a  $y \in \Lambda_N$  such that

$$(2.102) \quad \int_{\delta\Lambda_N} [\phi_N^2(x+y) + (\nabla\phi_N)^2(x+y)] dx \leq \frac{|\delta\Lambda_N|}{|\Lambda_N|} [A_N(\phi_N) + C_N(\phi_N)].$$

Let  $\tau\phi_N \in H^1(\mathbb{R}^d)$  be defined by

$$(2.103) \quad (\tau\phi_N)(x) = \begin{cases} \phi_N(x+y) & (x \in \Lambda_N) \\ \phi_N([x]_N + y) \left\{ N \left( 1 - \frac{|x|}{|[x]_N|} \right) + 1 \right\} & (x \in \Lambda_{N+1} \setminus \Lambda_N) \\ 0 & (x \notin \Lambda_{N+1}) \end{cases}$$

with  $[x]_N$  the radial projection of  $x$  onto  $\delta\Lambda_N$ , i.e.,  $\tau\phi_N$  linearly drops to 0 outside  $\Lambda_N$  along radial lines. Then, clearly,  $(\tau\phi_N)^2(x) \leq \phi_N^2([x]_N + y)$  and  $(\nabla\tau\phi_N)^2(x) \leq d(\nabla\phi_N)^2([x]_N + y)$  for all  $x \in \Lambda_{N+1} \setminus \Lambda_N$ . Hence, by (2.102),

$$(2.104) \quad \begin{aligned} A(\tau\phi_N) &\leq A_N(\phi_N) + \delta_N, & B(\tau\phi_N) &\leq B_N(\phi_N) + \kappa_a \delta_N, \\ C(\tau\phi_N) &\leq C_N(\phi_N) + \delta_N \end{aligned}$$

with

$$(2.105) \quad \delta_N = d \frac{|\delta\Lambda_{N+1}|}{|\delta\Lambda_N|} \frac{|\delta\Lambda_N|}{|\Lambda_N|} [A_N(\phi_N) + C_N(\phi_N)] = O\left(\frac{1}{N}\right).$$

Now define  $\phi^* \in H^1(\mathbb{R}^d)$  by

$$(2.106) \quad \phi^* = \frac{\tau\phi_N}{\sqrt{A(\tau\phi_N)}}.$$

Then clearly

$$(2.107) \quad A(\phi^*) = 1, \quad B(\phi^*) \leq B(\tau\phi_N), \quad C(\phi^*) = A(\tau\phi_N)C(\tau\phi_N),$$

where the second statement holds because  $A(\tau\phi_N) \geq A_N(\phi) = 1$ . Combining (2.101), (2.104) and (2.107), we get

$$(2.108) \quad A(\phi^*) = 1, \quad B(\phi^*) \leq b + \kappa_a \delta_N, \quad C(\phi^*) \leq (1 + \delta_N)[I_N^{\kappa_a}(b) + \varepsilon + \delta_N].$$

Hence we have

$$(2.109) \quad \begin{aligned} I^{\kappa_a}(b + \kappa_a \delta_N) &= \inf\{C(\phi): \phi \in H^1(\mathbb{R}^d), A(\phi) = 1, B(\phi) \leq b + \kappa_a \delta_N\} \\ &\leq C(\phi^*) \leq (1 + \delta_N)[I_N^{\kappa_a}(b) + \varepsilon + \delta_N]. \end{aligned}$$

Let  $N \rightarrow \infty$  and use (2.105) to get  $I^{\kappa_a}(b+) \leq \varepsilon + \liminf_{N \rightarrow \infty} I_N^{\kappa_a}(b)$ . Since  $\varepsilon > 0$  is arbitrary this proves the claim.

3. Combining Steps 1 and 2 and noting that  $b \mapsto I^{\kappa_a}(b)$  is right-continuous (because it is nonincreasing and lower semicontinuous), we have completed the proof of Proposition 2.  $\square$

### 3. The lower bound in Theorem 1

In this section we prove the complement of Propositions 1 and 2, which will complete the proof of Theorem 1. Recall from Section 2.1 that, by Brownian scaling,  $t^{-1}|W^a(t)|$  has the same distribution as  $|W^{a\tau^{-1/(d-2)}}(\tau)|$  with  $\tau = t^{(d-2)/d}$ .

**PROPOSITION 6.** *Let  $d \geq 3$  and  $a > 0$ . For every  $b > 0$ ,*

$$(3.1) \quad \liminf_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|W^{a\tau^{-1/(d-2)}}(\tau)| \leq b) \geq -I^{\kappa_a}(b),$$

where  $I^{\kappa_a}(b)$  is given by (1.7), (1.8).

*Proof.* Let  $C_N(\tau)$  be the event that the Brownian motion does not hit  $\partial\Lambda_{N-a}$  until time  $\tau$ . Clearly,

$$(3.2) \quad P(|W^{a\tau^{-1/(d-2)}}(\tau)| \leq b) \geq P(C_N(\tau), |W_N^{a\tau^{-1/(d-2)}}(\tau)| \leq b).$$

The right-hand side involves the Brownian motion on the torus, but restricted to stay a distance  $a$  away from the boundary. We can now simply repeat the argument in Section 2 on the event  $C_N(\tau)$ , the result being that

$$(3.3) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P\left(|W_N^{a\tau^{-1/(d-2)}}(\tau)| \leq b \mid C_N(\tau)\right) = -\tilde{I}_N^{\kappa_a}(b)$$

where  $\tilde{I}_N^{\kappa_a}(b)$  is given by the same formulas as in (1.7), (1.8), except that  $\mathbb{R}^d$  is replaced by  $\Lambda_N$  and  $\phi$  is restricted to  $\text{supp}(\phi) \cap \partial\Lambda_N = \emptyset$ . We have

$$(3.4) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(C_N(\tau)) = -\lambda_N$$

with  $\lambda_N$  the principal Dirichlet eigenvalue of  $-\Delta/2$  on  $\Lambda_N$ . Combining (3.2)–(3.4), we get

$$(3.5) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|W^{a\tau^{-1/(d-2)}}(\tau)| \leq b) \geq -\tilde{I}_N^{\kappa_a}(b) - \lambda_N \quad \text{for all } N.$$

Since  $\lim_{N \rightarrow \infty} \lambda_N = 0$ , it therefore suffices to show that

$$(3.6) \quad \lim_{N \rightarrow \infty} \tilde{I}_N^{\kappa_a}(b) = I^{\kappa_a}(b).$$

But this follows from the same type of argument as in Section 2.6.  $\square$

#### 4. Upper and lower bounds in Theorem 2

In this section we explain how the arguments given in Sections 2 and 3 for the Wiener sausage in  $d \geq 3$  can be carried over to  $d = 2$ . The necessary modifications are relatively minor and mainly involve a change in the choice of the scaling parameters.

*Upper bound.* 1. Fix  $N > 0$ . Wrap the Brownian motion around  $\Lambda_{N\sqrt{t/\log t}}$ , shrink space by  $\sqrt{t/\log t}$  and time by  $t/\log t$ . Then the analogue of (2.2), (2.3) reads

$$(4.1) \quad P(|W^a(t)| \leq bt/\log t) \leq P(|W_N^{a\sqrt{\tau e^{-\tau}}}(\tau)| \leq b) \text{ with } \tau = \log t.$$

We shall show how to obtain the analogue of Proposition 1, namely,

$$(4.2) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(|W_N^{a\sqrt{\tau e^{-\tau}}}(\tau)| \leq b) = -I_N^{2\pi}(b),$$

where  $I_N^{2\pi}(b)$  is given by the same formulas as in (1.7), (1.8), except that  $\mathbb{R}^d$  is replaced by  $\Lambda_N$  and  $\kappa_a$  by  $2\pi$ . Since, in Section 2.6, Proposition 2 was actually proved for any dimension, the claim in (4.2) will provide the upper bound in Theorem 2.

2. Henceforth we suppress the indices  $a, N$  and abbreviate

$$(4.3) \quad V_\tau = |W_N^{a\sqrt{\tau e^{-\tau}}}(\tau)|.$$

The analogue of Proposition 3 in Section 2.1 for  $d = 2$  reads:

**PROPOSITION 7.**  $(V_\tau)_{\tau > 0}$  satisfies the LDP on  $\mathbb{R}_+$  with rate  $\tau$  and with rate function

$$(4.4) \quad J_N^{2\pi}(b) = \inf_{\phi \in \partial\Phi_N^{2\pi}(b)} \left[ \frac{1}{2} \int_{\Lambda_N} |\nabla \phi|^2(x) dx \right]$$

with

$$(4.5) \quad \partial\Phi_N^{2\pi}(b) = \left\{ \phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x) dx = 1, \int_{\Lambda_N} \left( 1 - e^{-2\pi\phi^2(x)} \right) dx = b \right\}.$$

This is the same as Proposition 3, but with  $\kappa_a$  replaced by  $2\pi$ . To prove Proposition 7, the coarse-graining argument in Sections 2.2–2.4 can essentially be copied. All that we need to do is replace  $T_\tau$  defined in (2.11) everywhere by

$$(4.6) \quad T_\tau = \frac{1}{\tau e^{-\tau}}$$

and prove the technical lemmas.

3. Section 2.2 carries over with the following difference. On the right-hand side of (2.22) we end up with the expression

$$(4.7) \quad E\left(\exp\left[\frac{\tau}{\varepsilon T_\tau} |W^a(\varepsilon T_\tau)|\right]\right),$$

i.e., with an extra factor  $\tau$  in the exponent. Since  $\tau/\log T_\tau \rightarrow 1$  as  $\tau \rightarrow \infty$ , this means that instead of (2.23) we now need that

$$(4.8) \quad \sup_{T \geq 1} E\left(\exp\left[\frac{\log T}{T} |W^a(T)|\right]\right) < \infty.$$

However, this again follows from the results in van den Berg and Bolthausen [2]. Also, on the right-hand side of (2.38) we end up with the expression

$$(4.9) \quad E_{x\sqrt{\varepsilon T_\tau}, \varepsilon T_\tau}\left(\exp\left[\frac{\tau}{\varepsilon T_\tau} |W^a(\varepsilon T_\tau)|\right]\right),$$

i.e., again with the extra factor  $\tau$  in the exponent. This too can be accommodated because of (4.8).

4. Section 2.3 carries over after we prove Lemmas 2–4 for the new scaling in (4.6), with the following difference. We need to adapt the argument at the point where we are cutting out small holes around the points  $\beta(i\varepsilon)$ ,  $1 \leq i \leq \tau/\varepsilon$  (recall (2.47), (2.49)). Namely, this time we cut out holes of radius  $1/\sqrt{\log T_\tau \log \log T_\tau}$ , which is considerably larger than the radius  $K/\sqrt{T_\tau}$  used before. The total volume of the holes is at most  $(\tau/\varepsilon + 1)(\pi/\log T_\tau \log \log T_\tau)$ , which for  $\tau \rightarrow \infty$  tends to zero and therefore is negligible. The larger radius is needed for Part (a) of the new version of Lemma 2, which reads:

- LEMMA 8. (a)  $\lim_{\tau \rightarrow \infty} \sup_{y,z \notin B_{1/\sqrt{\log T_\tau \log \log T_\tau}}} q_{\tau,\varepsilon}(y,z) = 0$ .  
(b)  $\lim_{\tau \rightarrow \infty} \sup_{y,z \notin B_\rho} |\tau q_{\tau,\varepsilon}(y,z) - 2\pi\varphi_\varepsilon(y,z)| = 0$  for all  $0 < \rho < N/4$ .

*Proof.* (a) Step i of Part (a) in the proof of Lemma 2 carries over, so that (2.58) again applies. Step ii of Part (a) is replaced by the following argument. For any  $R > |y| > eb > 0$ ,

$$(4.10) \quad P_y^\infty(\sigma_b \leq \varepsilon/2) \leq P_y^\infty(\sigma_b < \hat{\sigma}_R) + P_y^\infty(\hat{\sigma}_R \leq \varepsilon/2),$$

where  $\hat{\sigma}_R$  is the first entrance time into  $B_R^c = \mathbb{R}^d \setminus B_R$ . We have

$$(4.11) \quad \begin{aligned} P_y^\infty(\sigma_b < \hat{\sigma}_R) &= \log\left(\frac{R}{|y|}\right) / \log\left(\frac{R}{b}\right), \\ P_y^\infty(\hat{\sigma}_R \leq \varepsilon/2) &\leq 4 \exp\left[-\frac{(R - |y|)^2}{\varepsilon}\right] \end{aligned}$$

(for the latter see e.g. van den Berg and Davies [3, Lemma 6.3]). The choice  $R = |y| + \sqrt{\varepsilon \log \log(|y|/b)}$  together with the inequality  $\log(1+x) \leq x$  for  $x \geq 0$

yields

$$(4.12) \quad P_y^\infty(\sigma_b \leq \varepsilon/2) \leq \frac{1}{\log(\frac{|y|}{b})} \left[ 4 + \frac{1}{|y|} \sqrt{\varepsilon \log \log \left( \frac{|y|}{b} \right)} \right].$$

Inserting this into (2.58), we get for any  $0 < eb < b' < N/4$ ,

$$(4.13) \quad \sup_{y,z \notin B_{b'}} q_b(y, z) \leq \frac{2c_2}{1 - c_3} \frac{1}{\log(\frac{b'}{b})} \left[ 4 + \frac{1}{b'} \sqrt{\varepsilon \log \log \left( \frac{b'}{b} \right)} \right].$$

Now put  $b = a/\sqrt{T_\tau}$ ,  $b' = 1/\sqrt{\log T_\tau \log \log T_\tau}$  and use that  $\log T_\tau \sim \tau$  ( $\tau \rightarrow \infty$ ), to get the claim.  $\square$

(b) Part (b) in the proof of Lemma 2 carries over, with the only difference that (2.64) is to be replaced by

$$(4.14) \quad \lim_{b \downarrow 0} \frac{1}{\pi / \log(\frac{1}{b})} P_y^\infty(\sigma_b \leq t) = \int_0^t p_s(-y) ds \quad \text{for all } y \in \mathbb{R}^2, t \geq 0$$

(Le Gall [16]). For  $b = a/\sqrt{T_\tau}$  we have  $\pi / \log(\frac{1}{b}) \sim 2\pi/\tau$  ( $\tau \rightarrow \infty$ ), which explains how the factor  $2\pi$  arises, replacing  $\kappa_a$ .  $\square$

Lemmas 3 and 4 were based on Lemma 2(b). It is obvious that with the new version in Lemma 8(b) the rest of the argument in Section 2.3 is unchanged.

5. Section 2.4 carries over verbatim with only  $\kappa_a$  to be replaced by  $2\pi$  everywhere. Section 2.5 also has no changes. In fact, in both these sections dimension plays no role at all.

*Lower bound.* The proof of Proposition 6 carries over after the appropriate changes in scaling.

## 5. Analysis of the variational problem

This section contains the main analytic part of our paper. Theorems 3(i)–(iii) are proved in Sections 5.1–5.3, Theorems 4(i), (ii) in Sections 5.4, 5.5, and Theorems 5(i)–(iii) in Sections 5.6–5.8. Recall the notation introduced in Section 1.

We will repeatedly make use of the following scaling relations. Let  $\phi \in H^1(\mathbb{R}^d)$ . For  $p, q > 0$ , define  $\psi \in H^1(\mathbb{R}^d)$  by

$$(5.1) \quad \psi(x) = q\phi(x/p).$$

Then

$$(5.2) \quad \begin{aligned} \|\nabla\psi\|_2^2 &= q^2 p^{d-2} \|\nabla\phi\|_2^2, \quad \|\psi\|_2^2 = q^2 p^d \|\phi\|_2^2, \quad \|\psi\|_4^4 = q^4 p^d \|\phi\|_4^4, \\ &\int (1 - e^{-\psi^2}) = p^d \int (1 - e^{-q^2\phi^2}). \end{aligned}$$

We will also repeatedly make use of the following Sobolev inequalities (see Lieb and Loss [19, pp. 186 and 190]):

$$(5.3) \quad S_d \|f\|_q^2 \leq \|\nabla f\|_2^2 \quad (d \geq 3, f \in D^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$$

with

$$(5.4) \quad q = \frac{2d}{d-2}, \quad S_d = d(d-2)2^{-2(d-1)/d}\pi^{(d+1)/d} \left[ \Gamma\left(\frac{d+1}{2}\right) \right]^{-2/d},$$

and

$$(5.5) \quad \|f\|_4 \leq S_{2,4} (\|\nabla f\|_2^2 + \|f\|_2^2)^{1/2} \quad (d=2, f \in H^1(\mathbb{R}^2))$$

with  $S_{2,4} = (4/27\pi)^{1/4}$ .

**5.1. Proof of Theorem 3(i), reduction to radially symmetric nonincreasing functions, and adaptation of the constraints.**

1. We begin by reformulating the variational problem for  $I^{\kappa_a}(b)$  in Theorem 1.

LEMMA 9. *Let  $d \geq 2$  and  $a > 0$ . For every  $b > 0$*

$$(5.6) \quad I^{\kappa_a}(b) = \frac{1}{2\kappa_a^{2/d}} \chi\left(\frac{b}{\kappa_a}\right),$$

where  $\chi: (0, \infty) \mapsto [0, \infty)$  is given by

$$(5.7) \quad \chi(u) = \inf \left\{ \|\nabla\psi\|_2^2 : \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) \leq u \right\}.$$

*Proof.* Apply (5.1) and (5.2) with  $p = \kappa_a^{-1/d}$  and  $q = \kappa_a^{1/2}$  to (1.7), (1.8).  $\square$

Lemma 9 proves Theorem 3(i).

2. The following lemma reduces the variational problem in (5.7) to radially symmetric nonincreasing (RSNI) functions. This reduction will become important later on.

LEMMA 10. *Let*

$$(5.8) \quad \mathcal{R}_u = \left\{ \psi \in H^1(\mathbb{R}^d) : \psi \text{ RSNI}, \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) \leq u \right\}.$$

Then

$$(5.9) \quad \chi(u) = \inf\{ \|\nabla\psi\|_2^2 : \psi \in \mathcal{R}_u \}.$$

*Proof.* It is clear that  $\chi(u) \leq \inf\{ \|\nabla\psi\|_2^2 : \psi \in \mathcal{R}_u \}$ . To prove the reverse, we let  $\psi^*$  denote the symmetric decreasing rearrangement of  $\psi$ . Then (see Lieb and Loss [19, §3.3 and §7.17])  $\psi^*$  is nonnegative, RSNI, and

$$(5.10) \quad \|\nabla\psi\|_2 \geq \|\nabla\psi^*\|_2, \quad \|\psi\|_2 = \|\psi^*\|_2, \quad \int(1 - e^{-\psi^2}) = \int(1 - e^{-\psi^{*2}}).$$

Hence

$$\begin{aligned} (5.11) \quad \chi(u) &\geq \inf\{ \|\nabla\psi^*\|_2^2 : \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \int(1 - e^{-\psi^2}) \leq u \} \\ &= \inf\{ \|\nabla\psi^*\|_2^2 : \psi^* \in H^1(\mathbb{R}^d), \|\psi^*\|_2 = 1, \int(1 - e^{-\psi^{*2}}) \leq u \} \\ &\geq \inf\{ \|\nabla\psi\|_2^2 : \psi \in \mathcal{R}_u \}. \end{aligned} \quad \square$$

3. The following lemma makes a statement about the minimisers of (5.7). Whether or not these exist will be established later on.

LEMMA 11. *Any minimiser of (5.7) is strictly positive, radially symmetric (modulo shifts) and strictly decreasing in the radial component.*

*Proof.* Let  $\psi$  be any minimiser of (5.7). Let  $\psi^*$  be its symmetric decreasing rearrangement. Then, by (5.10),  $\psi^*$  too is a minimiser of (5.7). By Brothers and Ziemer [9, Th. 1.1],  $\|\nabla\psi\|_2 > \|\nabla\psi^*\|_2$  if  $\psi$  is not a shift of  $\psi^*$  and the set  $\{x \in \mathbb{R}^d : (\nabla\psi^*)(x) = 0\}$  has zero Lebesgue measure. We will show that  $d\psi^*/dr < 0$ . Therefore  $\psi$  must be a shift of  $\psi^*$  (otherwise  $\psi$  could not be a minimiser) and the claim will follow.

Since  $\psi^*$  is a radially symmetric minimiser of (5.7), it satisfies the Euler-Lagrange equation

$$(5.12) \quad \frac{d^2\psi^*}{dr^2} + \frac{d-1}{r} \frac{d\psi^*}{dr} = \lambda\psi^*(1 - e^{-\psi^{*2}}) + \mu\psi^* \quad (r > 0),$$

where  $\lambda, \mu$  are Lagrange multipliers (see Berestycki and Lions [1, §5b]). By differentiating (5.12) repeatedly with respect to  $r$ , we see that  $\psi^* \in C^\infty(0, \infty)$ . Now, we already know that  $d\psi^*/dr \leq 0$ . Suppose that  $(d\psi^*/dr)(r_0) = 0$  for some  $r_0 > 0$ . Then clearly we must also have  $(d^2\psi^*/dr^2)(r_0) = 0$ . But from the derivatives of (5.12) it then follows that  $(d^n\psi^*/dr^n)(r_0) = 0$  for all  $n \in \mathbb{N}$ .

However, (5.12) is a second order differential equation with Lipschitz coefficients, and therefore the latter entails that  $\psi^*(r) = \psi^*(r_0)$  for all  $r \in (0, \infty)$ , i.e.,  $\psi^*$  is constant. But this contradicts  $\|\psi^*\|_2 = 1$ . Hence  $(d\psi^*/dr)(r_0) < 0$ . This proves the claim since  $r_0$  was arbitrary.  $\square$

4. We end this section with a lemma stating that the constraints in (5.7) can be adapted. This will turn out to be important later on.

LEMMA 12. *Let*

$$(5.13) \quad \begin{aligned} \widehat{\chi}(u) &= \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) = u\}, \\ \widetilde{\chi}(u) &= \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 \leq 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) = u\}. \end{aligned}$$

*Then*

$$(5.14) \quad \chi(u) = \widehat{\chi}(u) = \widetilde{\chi}(u).$$

*Proof.* We use an approximation argument.

i. It is clear from (5.7) and (5.13) that  $\chi(u) \leq \widehat{\chi}(u)$ . To prove the reverse, let  $(\psi_j)$  be a minimising sequence of  $\chi(u)$ . Then  $\|\psi_j\|_2 = 1$ ,  $\int(1 - e^{-\psi_j^2}) \leq u$ , and  $\|\nabla\psi_j\|_2^2 \rightarrow \chi(u)$  as  $j \rightarrow \infty$ . Define, for  $a > 0$ ,

$$(5.15) \quad g_\psi(a) = a^d \int (1 - e^{-a^{-d}\psi^2}).$$

Then

$$(5.16) \quad g'_\psi(a) = da^{d-1} \int (1 - e^{-a^{-d}\psi^2} - a^{-d}\psi^2 e^{-a^{-d}\psi^2}).$$

Since  $1 - e^{-x} - xe^{-x} \geq 0$  for  $x \geq 0$ , we have that  $g'_\psi(a) \geq 0$ . Since  $g_{\psi_j}(\infty) = \|\psi_j\|_2^2 = 1$  and  $g_{\psi_j}(1) = \int(1 - e^{-\psi_j^2}) \leq u$ , we see that there exists a sequence  $(a_j)$  with  $a_j \geq 1$  such that

$$(5.17) \quad g_{\psi_j}(a_j) = u \quad \text{for all } j.$$

Next, let  $\phi_j \in H^1(\mathbb{R}^d)$  be defined by  $\phi_j(x) = a_j^{-d/2}\psi_j(x/a_j)$ . Then, recalling (5.1), (5.2) and using (5.17), we see that

$$(5.18) \quad \begin{aligned} \|\nabla\phi_j\|_2^2 &= \frac{1}{a_j^2} \|\nabla\psi_j\|_2^2, \quad \|\phi_j\|_2^2 = \|\psi_j\|_2^2 = 1, \quad \int (1 - e^{-\phi_j^2}) = u \quad \text{for all } j. \end{aligned}$$

Hence (5.13) gives

$$(5.19) \quad \widehat{\chi}(u) \leq \|\nabla\phi_j\|_2^2 = \frac{1}{a_j^2} \|\nabla\psi_j\|_2^2 \leq \|\nabla\psi_j\|_2^2 \quad \text{for all } j.$$

But  $\|\nabla\psi_j\|_2^2 \rightarrow \chi(u)$  as  $j \rightarrow \infty$ , and so  $\widehat{\chi}(u) \leq \chi(u)$ .

ii. It is clear from (5.13) that  $\tilde{\chi}(u) \leq \hat{\chi}(u)$ . To prove the reverse, we begin with the following observation:

LEMMA 13. *The set*

$$(5.20) \quad \left\{ \psi \in H^1(\mathbb{R}^d): \psi \text{ RSNI}, \|\nabla \psi\|_2 \leq C, \|\psi\|_2 \leq 1, \int (e^{-\psi^2} - 1 + \psi^2) = 1 - u \right\}$$

*is a compact for all  $C < \infty$ .*

Before proving Lemma 13 we first complete the argument. Since  $\psi \mapsto \|\nabla \psi\|_2$  is lower semi-continuous, it follows from (5.20) that the variational problem for  $\tilde{\chi}(u)$  has a minimiser, say  $\psi^*$ . For  $n \in \mathbb{N}$ , let

$$(5.21) \quad p_n(x) = \frac{1}{\pi^{d/2} n^d} e^{-|x|^2/n^2} \quad (x \in \mathbb{R}^d),$$

and note that  $\int p_n = 1$  and  $\int (\nabla \sqrt{p_n})^2 = 2d/n^2$ . Now define  $\psi_n^*$  by

$$(5.22) \quad \psi_n^{*2} = \psi^{*2} + [1 - \|\psi^*\|_2^2] p_n.$$

Then  $\|\psi_n^*\|_2 = 1$  for all  $n$ . Moreover, since  $x \mapsto e^{-x} - 1 + x$  is increasing on  $[0, \infty)$ , we have

$$(5.23) \quad \int (e^{-\psi_n^{*2}} - 1 + \psi_n^{*2}) \geq \int (e^{-\psi^{*2}} - 1 + \psi^{*2}) = 1 - u \quad \text{for all } n.$$

Therefore  $\psi_n^*$  satisfies the constraints in the variational problem for  $\chi(u)$ , implying that

$$(5.24) \quad \chi(u) \leq \|\nabla \psi_n^*\|_2^2 \quad \text{for all } n.$$

But by the convexity inequality for gradients (Lieb and Loss [19, Th. 7.8]) we have

$$(5.25) \quad \|\nabla \psi_n^*\|_2^2 \leq \|\nabla \psi^*\|_2^2 + [1 - \|\psi^*\|_2^2] \|\nabla \sqrt{p_n}\|_2^2 = \tilde{\chi}(u) + [1 - \|\psi^*\|_2^2] \frac{2d}{n^2}.$$

Letting  $n \rightarrow \infty$ , we thus end up with  $\chi(u) \leq \tilde{\chi}(u)$ . But  $\chi(u) = \hat{\chi}(u)$  by Step i, and so the claim is proved.

iii. It thus remains to prove Lemma 13.

*Proof.* The key point is to show that the contribution to the integral in (5.20) coming from small  $x$  and from large  $x$  is uniformly small. First we pick  $0 < R < \infty$  and estimate

$$(5.26) \quad \int_{B_R^c} (e^{-\psi^2} - 1 + \psi^2) \leq \int_{B_R^c} \frac{1}{2} \psi^4 \leq \frac{1}{2\omega_d R^d} \int_{B_R^c} \psi^2 \leq \frac{1}{2\omega_d R^d},$$

where  $\omega_d$  is the volume of the ball with unit radius, and we use that  $\psi$  is RSNI and  $\|\psi\|_2 \leq 1$ . Next, we pick  $0 < r < \infty$  and estimate, using Hölder's

inequality,

$$(5.27) \quad \int_{B_r} (e^{-\psi^2} - 1 + \psi^2) \leq \int_{B_r} \psi^2 \leq (\omega_d r^d)^{1/p} \|\psi\|_{2q}^2 \quad \left( p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1 \right).$$

The last factor may be estimated with the help of the Sobolev inequalities in (5.3)–(5.5):

$$(5.28) \quad \begin{aligned} \|\psi\|_{2d/(d-2)}^2 &\leq S_d \|\nabla \psi\|_2^2 & (d \geq 3) \\ \|\psi\|_4^2 &\leq S_{2,4} (\|\nabla \psi\|_2^2 + \|\psi\|_2^2) & (d = 2). \end{aligned}$$

Thus, picking  $p = d/2, q = d/(d-2)$  for  $d \geq 3$  and  $p = q = 2$  for  $d = 2$ , we obtain using  $\|\nabla \psi\|_2 \leq C$  that

$$(5.29) \quad \int_{B_r} (e^{-\psi^2} - 1 + \psi^2) \leq \begin{cases} C_d r & (d \geq 3) \\ C_2 r^{1/2} & (d = 2). \end{cases}$$

We see from (5.26) and (5.29) that the contribution from  $B_R^c$  and  $B_r$  tends to zero uniformly in  $\psi$  as  $R \rightarrow \infty$  and  $r \downarrow 0$ . We can now complete the proof as follows. Any sequence  $(\psi_j)$  in  $H^1(\mathbb{R}^d)$  has a subsequence that converges to some  $\psi \in H^1(\mathbb{R}^d)$  uniformly on every annulus  $B_R \setminus B_r$  (use the fact that  $\psi_j$  is RSNI and  $\|\psi_j\|_2 \leq 1$  for all  $j$ ). Because  $\psi_j$  is RSNI,  $\|\nabla \psi_j\|_2 \leq C$ ,  $\|\psi_j\|_2 \leq 1$  for all  $j$ , the same is true for  $\psi$ . Moreover, since

$$(5.30) \quad \begin{aligned} \int (e^{-\psi_j^2} - 1 + \psi_j^2) &= 1 - u \quad \text{for all } j, \\ \lim_{j \rightarrow \infty} \int_{B_R \setminus B_r} (e^{-\psi_j^2} - 1 + \psi_j^2) &= \int_{B_R \setminus B_r} (e^{-\psi^2} - 1 + \psi^2), \\ \lim_{r \downarrow 0, R \rightarrow \infty} \int_{B_R \setminus B_r} (e^{-\psi^2} - 1 + \psi^2) &= \int (e^{-\psi^2} - 1 + \psi^2), \end{aligned}$$

we also have  $\int (e^{-\psi^2} - 1 + \psi^2) = 1 - u$ . Therefore  $\psi$  is in the set.  $\square$

This completes the proof of Lemma 13 and hence of Lemma 12.  $\square$

The reason behind Lemma 13 is the following. Although  $\psi$  may lose  $L^2$ -mass to infinity, the integral cannot. Indeed, following an argument in Brezis and Lieb [8], we can show that if  $\|\psi\|_2^2$  loses mass  $\rho \in (0, u]$ , then also  $\int (1 - e^{-\psi^2})$  loses mass  $\rho$ , and so  $\int (e^{-\psi^2} - 1 + \psi^2)$  loses nothing.

In the sequel we shall often suppress the condition  $\psi \in H^1(\mathbb{R}^d)$  from the notation.

### 5.2. Proof of Theorem 3(ii).

1. Since  $1 - e^{-x} \leq x$  for  $x \geq 0$ , we have  $\int (1 - e^{-\psi^2}) \leq \|\psi\|_2^2$ . So, for  $u \geq 1$ , (5.7) reduces to

$$(5.31) \quad \chi(u) = \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1 \}.$$

Suppose  $\psi \in H^1(\mathbb{R}^d)$  is such that  $\|\psi\|_2 = 1$ . Apply to (5.31) the scaling in (5.1) and (5.2) with  $p > 0$  arbitrary and  $q = p^{-d/2}$ , to obtain  $\chi(u) \leq p^{-2}\|\nabla\psi\|_2^2$  for  $u \geq 1$ . Taking the limit  $p \rightarrow \infty$ , we get  $\chi(u) = 0$  for  $u \geq 1$ .

2. It follows from Theorems 4(ii) and 5(iii) that  $\chi$  is strictly positive in a left-neighbourhood of 0. Since, by Theorem 3(iii),  $u \mapsto u^{2/d}\chi(u)$  is nonincreasing on  $(0, 1)$ , it follows that  $\chi$  is strictly decreasing on  $(0, 1)$ .

3. Step 1 shows that  $\chi$  is continuous on  $(1, \infty)$ . Theorems 4(ii) and 5(iii) imply that  $\chi$  is continuous at  $u = 1$ . Therefore we need only prove continuity on  $(0, 1)$ . Let  $u_0 \in (0, 1)$  be arbitrary. Since  $\chi$  is lower semi-continuous and nonincreasing, it is right-continuous. Let

$$(5.32) \quad \delta = \lim_{u \uparrow u_0} \chi(u) - \chi(u_0) \geq 0.$$

We shall show that  $\delta = 0$  by using a perturbation argument.

4. Let  $\varepsilon > 0$  be arbitrary. Then, because  $\chi(u) = \hat{\chi}(u)$  by Lemma 12, there exist  $\psi_\varepsilon, \Phi_\varepsilon \in H^1(\mathbb{R}^d)$  satisfying

$$(5.33) \quad \begin{aligned} \|\psi_\varepsilon\|_2 &= 1, \int (1 - e^{-\psi_\varepsilon^2}) = u_0, & \|\nabla\psi_\varepsilon\|_2^2 &\leq \chi(u_0) + \varepsilon, \\ \|\Phi_\varepsilon\|_2 &= 1, \int (1 - e^{-\Phi_\varepsilon^2}) = u_0 - \varepsilon, & \|\nabla\Phi_\varepsilon\|_2^2 &\leq \chi(u_0 - \varepsilon) + \varepsilon. \end{aligned}$$

Define, for  $0 \leq \alpha \leq 1$ ,

$$(5.34) \quad \Lambda_{\alpha,\varepsilon} = [\alpha\psi_\varepsilon^2 + (1 - \alpha)\Phi_\varepsilon^2]^{1/2}.$$

Then, by (5.33),

$$(5.35) \quad \|\Lambda_{\alpha,\varepsilon}\|_2^2 = \int [\alpha\psi_\varepsilon^2 + (1 - \alpha)\Phi_\varepsilon^2] = 1$$

and, by the convexity inequality for gradients (see Lieb and Loss [19, Th. 7.8]),

$$(5.36) \quad \begin{aligned} \|\nabla\Lambda_{\alpha,\varepsilon}\|_2^2 &\leq \alpha\|\nabla\psi_\varepsilon\|_2^2 + (1 - \alpha)\|\nabla\Phi_\varepsilon\|_2^2 \\ &\leq \alpha(\chi(u_0) + \varepsilon) + (1 - \alpha)(\chi(u_0 - \varepsilon) + \varepsilon) \\ &= \alpha\chi(u_0) + (1 - \alpha)\chi(u_0 - \varepsilon) + \varepsilon. \end{aligned}$$

Next define, for  $0 \leq \alpha \leq 1$ ,

$$(5.37) \quad k(\alpha) = \int (1 - e^{-\Lambda_{\alpha,\varepsilon}^2}).$$

Then, by (5.34),

$$(5.38) \quad k''(\alpha) = \int (\psi_\varepsilon^2 - \Phi_\varepsilon^2)^2 e^{-\alpha\psi_\varepsilon^2 - (1 - \alpha)\Phi_\varepsilon^2}.$$

It follows that  $k$  is convex on  $[0, 1]$ , and consequently

$$(5.39) \quad k(\alpha) \leq \alpha k(1) + (1 - \alpha)k(0) = \alpha u_0 + (1 - \alpha)(u_0 - \varepsilon).$$

By the convexity of  $k$  and by (5.39), there exists a unique  $\alpha_\varepsilon \in [\frac{1}{2}, 1)$  such that  $k(\alpha_\varepsilon) = u_0 - \varepsilon/2$ . By (5.35)–(5.37), Lemma 12 and the fact that  $\chi$  is nonincreasing, we therefore have

$$\begin{aligned} (5.40) \quad \chi(u_0 - \varepsilon/2) &= \widehat{\chi}(u_0 - \varepsilon/2) \leq \|\nabla \Lambda_{\alpha_\varepsilon, \varepsilon}\|_2^2 \\ &\leq \alpha_\varepsilon \chi(u_0) + (1 - \alpha_\varepsilon) \chi(u_0 - \varepsilon) + \varepsilon \\ &\leq \frac{1}{2} \chi(u_0) + \frac{1}{2} \chi(u_0 - \varepsilon) + \varepsilon. \end{aligned}$$

Hence

$$(5.41) \quad \lim_{\varepsilon \downarrow 0} \chi(u_0 - \varepsilon/2) \leq \frac{1}{2} \chi(u_0) + \frac{1}{2} \lim_{\varepsilon \downarrow 0} \chi(u_0 - \varepsilon).$$

Combining (5.32) and (5.41), we therefore arrive at

$$(5.42) \quad \chi(u_0) + \delta \leq \frac{1}{2} \chi(u_0) + \frac{1}{2} [\chi(u_0) + \delta].$$

Thus  $\delta = 0$  and  $\chi$  is continuous at  $u_0$ . This proves the continuity of  $\chi$  on  $(0, 1)$ , since  $u_0 \in (0, 1)$  was arbitrary.

### 5.3. Proof of Theorem 3(iii).

1. The first claim in Theorem 3(iii) is proved as follows. Apply (5.1), (5.2) with  $p = u^{1/d}$  and  $q = u^{-1/2}$  to (5.7), to obtain

$$(5.43) \quad u^{2/d} \chi(u) = \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \int (1 - e^{-u^{-1}\psi^2}) \leq 1 \}.$$

Since the integrand is nonincreasing in  $u$ , so is the infimum. Therefore we find that  $u \mapsto u^{2/d} \chi(u)$  is nonincreasing. To prove the strict monotonicity claimed in Theorem 3(iii), we need to wait until the end of Section 5.8, as this will require the existence of a minimiser for a certain range of  $u$ -values.

2. The second claim in Theorem 3(iii) is proved by deriving upper and lower bounds. Pick  $\delta > 0$ . Let  $B$  be any open set in  $\mathbb{R}^d$  with  $|B| = 1$ . Then, since  $1 - e^{-\delta^{-1}\psi^2} \geq 0$ , it follows from (5.43) that

$$\begin{aligned} (5.44) \quad \delta^{2/d} \chi(\delta) &\leq \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\delta^{-1}\psi^2}) \leq 1, \text{supp}(\psi) \subset B \right\} \\ &= \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \int_B (1 - e^{-\delta^{-1}\psi^2}) \leq 1, \text{supp}(\psi) \subset B \right\} \\ &\leq \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \text{supp}(\psi) \subset B \}. \end{aligned}$$

Take the infimum over  $B$  of the right-hand side of (5.44). This infimum equals  $\lambda_d$  by the Faber-Krahn isoperimetric inequality for the Dirichlet Laplacian (see Faris [14]). Hence  $\delta^{2/d} \chi(\delta) \leq \lambda_d$ , which proves the upper bound.

3. The lower bound is more laborious. Let  $\psi$  be a minimiser of (5.43). (In Sections 5.4 and 5.7 we will prove the existence of a minimiser of (5.7), and hence of (5.43), for  $u$  in a neighbourhood of 0 for any  $d \geq 1$ .) We exploit the radial symmetry and monotonicity of  $\psi$  established in Lemma 11. For  $t > 0$ , define the ball  $B_t = \{x \in \mathbb{R}^d : \psi^2(x) > t\}$  and put  $\mu(t) = |B_t|$ . Then

$$(5.45) \quad 1 = \int_{\mathbb{R}^d} \psi^2 = \int_{[0, \infty)} t d[-\mu(t)].$$

Moreover, abbreviating  $\varepsilon = \sqrt{\delta}$  we have

$$\begin{aligned} (5.46) \quad 1 \geq \int_{\mathbb{R}^d} (1 - e^{-\delta^{-1}\psi^2}) &= \int_{[0, \infty)} (1 - e^{-\delta^{-1}t}) d[-\mu(t)] \\ &\geq \int_{[\varepsilon, \infty)} (1 - e^{-\delta^{-1}t}) d[-\mu(t)] \\ &\geq (1 - e^{-1/\varepsilon}) \int_{[\varepsilon, \infty)} d[-\mu(t)] = (1 - e^{-1/\varepsilon}) |B_\varepsilon|. \end{aligned}$$

Hence

$$\begin{aligned} (5.47) \quad \int_{\mathbb{R}^d \setminus B_\varepsilon} \psi^2 &= \int_{[0, \varepsilon)} t d[-\mu(t)] \\ &\leq \frac{\varepsilon}{1 - e^{-1/\varepsilon}} \int_{[0, \varepsilon)} (1 - e^{-\delta^{-1}t}) d[-\mu(t)] \leq \frac{\varepsilon}{1 - e^{-1/\varepsilon}}, \end{aligned}$$

where we use the first inequality in (5.46). Combining (5.45) and (5.47) we obtain

$$(5.48) \quad \int_{B_\varepsilon} \psi^2 \geq 1 - \frac{\varepsilon}{1 - e^{-1/\varepsilon}}.$$

Next, define  $\zeta = \psi - \sqrt{\varepsilon}$ . Then  $\zeta > 0$  on  $B_\varepsilon$  and  $\zeta = 0$  on  $\partial B_\varepsilon$ . By Cauchy-Schwarz and  $\|\psi\|_2 = 1$ , we have

$$(5.49) \quad \int_{B_\varepsilon} \zeta \leq \int_{B_\varepsilon} \psi \leq \left( \int_{B_\varepsilon} \psi^2 \right)^{1/2} |B_\varepsilon|^{1/2} \leq |B_\varepsilon|^{1/2}.$$

Hence

$$(5.50) \quad \int_{B_\varepsilon} \psi^2 = \int_{B_\varepsilon} [\zeta + \sqrt{\varepsilon}]^2 \leq \int_{B_\varepsilon} \zeta^2 + 2\sqrt{\varepsilon}|B_\varepsilon|^{1/2} + \varepsilon|B_\varepsilon|.$$

Combining (5.46), (5.48), (5.50) and abbreviating  $\eta = \varepsilon/(1 - e^{-1/\varepsilon})$ , we obtain

$$(5.51) \quad \int_{B_\varepsilon} \zeta^2 \geq 1 - 2\sqrt{\eta} - 2\eta.$$

Finally, if we define  $\phi$  by  $\zeta = (1 - 2\sqrt{\eta} - 2\eta)^{1/2}\phi$ , then  $\int_{B_\varepsilon} \phi^2 \geq 1$  and  $\phi|_{\partial B_\varepsilon} = 0$ . So, recalling that  $\psi$  is a minimiser of (5.43), we get

$$(5.52) \quad \delta^{2/d} \chi(\delta) = \int_{\mathbb{R}^d} |\nabla \psi|^2 \geq \int_{B_\varepsilon} |\nabla \psi|^2 = \int_{B_\varepsilon} |\nabla \zeta|^2 = (1 - 2\sqrt{\eta} - 2\eta) \int_{B_\varepsilon} |\nabla \phi|^2$$

with

$$(5.53) \quad \begin{aligned} \int_{B_\varepsilon} |\nabla \phi|^2 &\geq \inf \left\{ \int_{B_\varepsilon} |\nabla \phi|^2 : \int_{B_\varepsilon} \phi^2 \geq 1, \phi|_{\partial B_\varepsilon} = 0 \right\} \\ &= \lambda_d(B_\varepsilon) = |B_\varepsilon|^{-2/d} \lambda_d \geq (1 - e^{-1/\varepsilon})^{2/d} \lambda_d, \end{aligned}$$

where we use the scaling of the smallest Dirichlet eigenvalue of  $-\Delta$  on  $B_\varepsilon$ , in combination with (5.46). Letting  $\delta \downarrow 0$  in (5.52) and using the fact that  $\varepsilon \downarrow 0$  and  $\eta \downarrow 0$ , we arrive at  $\liminf_{\delta \downarrow 0} \delta^{2/d} \chi(\delta) \geq \lambda_d$ .

**5.4. Proof of Theorem 4(i).** To prove that for  $2 \leq d \leq 4$  the variational problem in (5.7) has a minimiser for all  $u \in (0, 1)$ , we do a variational argument that takes advantage of Lemma 12. Fix  $u \in (0, 1)$ . Let  $\psi^*$  be any minimiser of  $\tilde{\chi}(u)$ , which we know exists by (5.20), i.e.,

$$(5.54) \quad \|\nabla \psi^*\|_2^2 = \tilde{\chi}(u), \quad \|\psi^*\|_2^2 \leq 1, \quad \int(e^{-\psi^{*2}} - 1 + \psi^{*2}) = 1 - u.$$

There are two cases. Either  $\|\psi^*\|_2 = 1$ , in which case  $\psi^*$  is also a minimiser of  $\chi(u)$  and we are done, or  $\|\psi^*\|_2 < 1$ . It remains to exclude the latter case.

$d = 2$ : Suppose that  $\|\psi^*\|_2^2 = 1 - \rho$  ( $\rho > 0$ ). Let  $\phi^*(\cdot) = \psi^*(\cdot(1 - \rho)^{1/2})$ . Then, by (5.1), (5.2),

$$(5.55) \quad \|\nabla \phi^*\|_2^2 = \|\nabla \psi^*\|_2^2 = \chi(u), \quad \|\phi^*\|_2^2 = 1, \quad \int(e^{-\phi^{*2}} - 1 + \phi^{*2}) = \frac{1-u}{1-\rho}.$$

Hence

$$(5.56) \quad \chi(u) = \|\nabla \phi^*\|_2^2 \geq \chi\left(\frac{u-\rho}{1-\rho}\right).$$

But  $(u - \rho)/(1 - \rho) < u$ , and so the right-hand side is strictly larger than  $\chi(u)$  by Theorem 3(ii), which is a contradiction.

$d = 3, 4$ : We can do smooth perturbations of  $\psi^*$  inside the class

$$\{\psi \in H^1(\mathbb{R}^d) : \|\psi\|_2 \leq 1, \int(e^{-\psi^2} - 1 + \psi^2) = 1 - u\}$$

to conclude that  $\psi^*$  must satisfy the Euler-Lagrange equation associated with the variational problem

$$(5.57) \quad \inf\{\|\nabla \psi\|_2^2 : \int(e^{-\psi^2} - 1 + \psi^2) = 1 - u\}.$$

But then we have a contradiction to the following:

**LEMMA 14.** *For  $d = 3, 4$  all solutions of the Euler-Lagrange equation associated with the variational problem*

$$(5.58) \quad \inf\{\|\nabla \psi\|_2^2 : \int(e^{-\psi^2} - 1 + \psi^2) = 1\}$$

*have infinite  $L^2$ -norm.*

*Proof.* By the results of Section 5b in Berestycki and Lions [1], there exists a Lagrange multiplier  $\lambda_d > 0$  such that

$$(5.59) \quad \Delta\psi = -\lambda_d\psi(1 - e^{-\psi^2}),$$

which is the Euler-Lagrange equation associated with the variational problem in (5.58). By the results of Section 5c in the same paper, we have  $\psi \in C^2(\mathbb{R}^d)$ . Since  $\psi$  is RSNI (recall Lemmas 10, 11), it follows from (5.58) that  $\psi \in L^4(\mathbb{R}^d)$ . Suppose that  $\psi \in L^2(\mathbb{R}^d)$ . Then, by Hölder's inequality,  $\psi \in L^3(\mathbb{R}^d)$ . Abbreviate the right-hand side of (5.59) by  $f_\psi$ . Then  $f_\psi \in L^1(\mathbb{R}^d)$ , and so we have

$$(5.60) \quad \psi = f_\psi * K,$$

where  $*$  denotes convolution and  $K$  is the Green function. It follows that

$$(5.61) \quad \int \psi^2 = \int dy_1 f(y_1) \int dy_2 f(y_2) \left[ \int dx K(x - y_1)K(x - y_2) \right].$$

But the last integral is infinite for all  $y_1, y_2 \in \mathbb{R}^d$  when  $d \leq 4$ , which is a contradiction.  $\square$

This proves the existence of a minimiser of (5.7) for  $2 \leq d \leq 4$ . The remaining claims in Theorem 4(i) follow from Lemma 11.

### 5.5. Proof of Theorem 4(ii).

1. To prove the first claim in Theorem 4(ii), we apply (5.1), (5.2) with  $p = (1 - u)^{-1/d}$  and  $q = (1 - u)^{1/2}$  to (5.7) to get

$$(5.62) \quad \begin{aligned} \chi(u) &= \inf \left\{ \|\nabla\psi\|_2^2 : \|\psi\|_2 = 1, \int (e^{-\psi^2} - 1 + \psi^2) \geq 1 - u \right\} \\ &= (1 - u)^{2/d} \inf \left\{ \|\nabla\psi\|_2^2 : \|\psi\|_2 = 1, \right. \\ &\quad \left. \int \left( \frac{e^{-(1-u)\psi^2} - 1 + (1-u)\psi^2}{(1-u)^2} \right) \geq 1 \right\}. \end{aligned}$$

Since the integrand is nondecreasing in  $u$ , it follows that the infimum is nonincreasing in  $u$ . Hence  $u \mapsto (1 - u)^{-2/d}\chi(u)$  is nonincreasing. To get strict monotonicity we use the existence of a minimiser as established in Section 5.4.

2. Let  $\psi^*$  be any minimiser of the variational problem in (5.62). Pick  $v \in (u, 1)$ . Then there exists a  $\delta_{u,v} > 0$  such that

$$(5.63) \quad \int \left( \frac{e^{-(1-v)\psi^{*2}} - 1 + (1-v)\psi^{*2}}{(1-v)^2} \right) \geq 1 + \delta_{u,v}.$$

Hence

$$\begin{aligned}
(5.64) \quad & (1-u)^{-2/d} \chi(u) = \|\nabla \psi^*\|_2^2 \\
& \geq \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \int \left( \frac{e^{-(1-v)\psi^2} - 1 + (1-v)\psi^2}{(1-v)^2} \right) \geq 1 + \delta_{u,v} \right\} \\
& = (1-v)^{-2/d} \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \right. \\
& \quad \left. \int (1 - e^{-\psi^2}) \leq 1 - (1 + \delta_{u,v})(1-v) \right\},
\end{aligned}$$

where we reverse the scaling that led from (5.7) to (5.62). But, the right-hand side is equal to  $(1-v)^{-2/d} \chi(v - \delta_{u,v}(1-v))$ , which is strictly larger than  $(1-v)^{-2/d} \chi(v)$  by Theorem 3(ii). This proves the strict monotonicity.

3. The second claim in Theorem 4(ii) is proved by deriving upper and lower bounds. Since  $e^{-x} \leq 1 - x + \frac{1}{2}x^2$  for  $x \geq 0$ , we have

$$(5.65) \quad \int (e^{-\psi^2} - 1 + \psi^2) \leq \frac{1}{2} \int_{\mathbb{R}^d} \psi^4.$$

Hence (5.7) gives

$$(5.66) \quad \chi(1-\delta) \geq \inf \left\{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \int \psi^4 \geq 2\delta \right\}.$$

We use (5.1), (5.2) with  $p = (2\delta)^{-1/d}$  and  $q = 1$  in (5.66), to obtain

$$(5.67) \quad \chi(1-\delta) \geq (2\delta)^{2/d} \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \|\psi\|_4 \geq 1 \} = (2\delta)^{2/d} \mu_d.$$

The fact that the infimum equals  $\mu_d$  in (1.14) follows from the same type of argument as in the proof of Lemma 12, showing that the constraint  $\|\psi\|_4 \geq 1$  may be replaced by  $\|\psi\|_4 = 1$ . For the case  $d = 4$  we shall see in Step 4 in Section 5.5 that the constraint  $\|\psi\|_4 = 1$  can even be dropped. Hence we have  $\delta^{-2/d} \chi(1-\delta) \geq 2^{2/d} \mu_d$ , which proves the lower bound.

4. Since  $e^{-\psi^2} - 1 + \psi^2 \geq \frac{1}{2}\psi^4 - \frac{1}{6}\psi^6$ , we have by (5.7),

$$(5.68) \quad \chi(1-\delta) \leq \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \|\psi\|_4^4 \geq 2\delta + \frac{1}{3}\|\psi\|_6^6 \}.$$

We apply (5.1), (5.2) with  $p = (2\delta)^{-1/d}$  and  $q = (2\delta)^{1/2}$ , to get

$$(5.69) \quad (2\delta)^{-2/d} \chi(1-\delta) \leq \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \|\psi\|_4^4 \geq 1 + \frac{2\delta}{3}\|\psi\|_6^6 \}.$$

$d=2$ : Insert the Sobolev inequality  $\|\psi\|_6 \leq S_{2,6}(1 + \|\nabla \psi\|_2^2)^{1/2}$  (Lieb and Loss [19, p. 190]) into (5.69), to get

$$\begin{aligned}
(5.70) \quad & (2\delta)^{-1} \chi(1-\delta) \leq \inf \{ \|\nabla \psi\|_2^2 : \|\psi\|_2 = 1, \|\psi\|_4^4 \geq 1 + \frac{2\delta}{3}S_{2,6}^6(1 + \|\nabla \psi\|_2^2)^3 \}.
\end{aligned}$$

By considering the trial function  $\psi_a(x) = (\pi a^2)^{-1/2} \exp[-|x|^2/2a^2]$  ( $x \in \mathbb{R}^2$ ) for  $a > 0$  sufficiently small, we see that the second constraint in (5.70) is satisfied for  $\delta > 0$  sufficiently small. Hence  $\|\nabla\psi\|_2^2 \leq M$  for some  $M < \infty$  and all  $\psi$  satisfying the constraints in (5.70). Thus, for all  $\delta > 0$  sufficiently small,

$$(5.71) \quad \begin{aligned} (2\delta)^{-1}\chi(1-\delta) &\leq \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \|\psi\|_4^4 \geq 1 + \frac{2\delta}{3}S_{2,6}^6(1+M)^3\} \\ &\leq (1 + \frac{2\delta}{3}S_{2,6}^6(1+M)^3)\mu_2, \end{aligned}$$

where we use (5.1), (5.2) once more, this time with  $p = (1 + \frac{2\delta}{3}S_{2,6}^6(1+M)^3)^{1/2}$  and  $q = p^{-1}$ . We also recall (1.14). Let  $\delta \downarrow 0$  to get the desired upper bound.

d = 3: Insert the Sobolev inequality  $\|\psi\|_6 \leq S_3^{-1/2}\|\nabla\psi\|_2$  (recall (5.3) and (5.4)) into (5.69), to get

$$(5.72) \quad (2\delta)^{-2/3}\chi(1-\delta) \leq \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \|\psi\|_4^4 \geq 1 + \frac{2\delta\|\nabla\psi\|_2^6}{3S_3^3}\}.$$

This time we use the trial function  $\psi_a(x) = (\pi a^2)^{-3/4} \exp[-|x|^2/2a^2]$  ( $x \in \mathbb{R}^3$ ), to see that the second constraint in (5.72) is satisfied for  $\delta > 0$  sufficiently small, and that  $\|\nabla\psi\|_2^2 \leq M$  for some  $M < \infty$  and all  $\psi$  satisfying the constraints in (5.72). After scaling we get

$$(5.73) \quad (2\delta)^{-2/3}\chi(1-\delta) \leq (1 + \frac{2\delta M^3}{3S_3^3})^{2/3}\mu_3,$$

and hence the desired upper bound after letting  $\delta \downarrow 0$ .

d = 4: This case is slightly more subtle.

$$(5.74) \quad \begin{aligned} \text{i. For any } M > 0 \text{ we may insert the constraint } \|\psi\|_\infty \leq M, \text{ to get} \\ (2\delta)^{-1/2}\chi(1-\delta) &\leq \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \|\psi\|_\infty \leq M, \|\psi\|_4^4 \geq 1 + \frac{2\delta M^2}{3}\|\psi\|_4^4\}. \end{aligned}$$

For any  $0 < \delta < 3/2M^2$  we therefore have

$$(5.75) \quad \begin{aligned} (2\delta)^{-1/2}\chi(1-\delta) &\leq \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \|\psi\|_\infty \leq M, \|\psi\|_4^4 \geq \left(1 - \frac{2\delta M^2}{3}\right)^{-1}\} \\ &= \left(1 - \frac{2\delta M^2}{3}\right)^{-1/2} \inf\{\|\nabla\psi\|_2^2: \|\psi\|_2 = 1, \\ &\quad \|\psi\|_\infty \leq M \left(1 - \frac{2\delta M^2}{3}\right)^{1/2}, \|\psi\|_4^4 \geq 1\}. \end{aligned}$$

We will construct a sequence  $(\psi_j)$  in  $D^1(\mathbb{R}^4)$  satisfying the constraints in (5.75), with  $\|\psi_j\|_4 = 1$  for all  $j$ , such that  $\limsup_{j \rightarrow \infty} \|\nabla \psi_j\|_2^2 \leq S_4$ . Since  $\mu_4 \geq S_4$ , this sequence will be a minimising sequence of  $\mu_4$  (recall (1.14) and (5.3)).

ii. Let  $\psi_0$  be defined by

$$(5.76) \quad \psi_0(x) = \left( |x|^2 + \sqrt{\pi^2/6} \right)^{-1} \quad (x \in \mathbb{R}^4).$$

We have  $\|\psi_0\|_4 = 1$ . For  $\alpha > 0$ , let  $\psi_\alpha$  be defined by

$$(5.77) \quad \psi_\alpha(x) = c_\alpha e^{-\alpha|x|} \psi_0(x) \quad (x \in \mathbb{R}^4),$$

where  $c_\alpha > 0$  is chosen such that  $\|\psi_\alpha\|_4 = 1$ . Consider now the scaling

$$(5.78) \quad \psi_{\alpha,\beta}(x) = \beta^{-1} \psi_\alpha(x/\beta) \quad (x \in \mathbb{R}^4)$$

with  $\beta = \beta(\alpha) > 0$  chosen such that  $\|\psi_{\alpha,\beta}\|_2 = 1$ , i.e.,  $\beta = \|\psi_\alpha\|_2^{-1}$ . We have  $\|\psi_{\alpha,\beta}\|_4 = 1$ , while

$$(5.79) \quad \|\psi_{\alpha,\beta}\|_\infty = c_\alpha \beta^{-1} \|\psi_0\|_\infty = c_\alpha \|\psi_\alpha\|_2 \sqrt{6/\pi^2}$$

with

$$(5.80)$$

$$\|\psi_\alpha\|_2^2 = c_\alpha^2 \int e^{-2\alpha|x|} \psi_0^2(x) dx \leq c_\alpha^2 \int_0^\infty e^{-2\alpha r} r^{-7/2} [2\pi^2 r^3] dr \leq C c_\alpha^2 \alpha^{-1/2}.$$

Hence  $\|\psi_{\alpha,\beta}\|_\infty \leq C' c_\alpha^2 \alpha^{-1/4}$ . Now we pick  $\alpha = \delta$  and  $M = \delta^{-1/3}$ , and we note that  $C' c_\delta^2 \delta^{-1/4} \leq \delta^{-1/3} (1 - \frac{2\delta^{1/3}}{3})^{1/2}$  for  $\delta$  sufficiently small because  $\lim_{\delta \downarrow 0} c_\delta = 1$ . Then  $\psi_{\alpha,\beta}$  satisfies the constraints in (5.75), and so for  $\delta$  sufficiently small,

$$(5.81) \quad (2\delta)^{-1/2} \chi(1 - \delta) \leq (1 - \frac{2\delta^{1/3}}{3})^{-1/2} \|\nabla \psi_{\alpha,\beta}\|_2^2 \quad \text{with } \alpha = \delta, \beta = \beta(\delta).$$

iii. It thus remains to show that  $\psi_{\alpha,\beta}$  is a minimising sequence for the variational problem defining  $\mu_4$ . To that end we first note that  $\|\nabla \psi_0\|_2^2 = S_4$ . Therefore, using (5.77) and (5.80), we have

$$\begin{aligned} (5.82) \quad \|\nabla \psi_{\alpha,\beta}\|_2^2 &= \|\nabla \psi_\alpha\|_2^2 \\ &= \alpha^2 \|\psi_\alpha\|_2^2 + c_\alpha^2 \int e^{-2\alpha|x|} |\nabla \psi_0(x)|^2 dx \\ &\quad + 2\alpha c_\alpha \int \psi_\alpha(x) e^{-\alpha|x|} |\nabla \psi_0(x)| dx \\ &\leq \alpha^2 \|\psi_\alpha\|_2^2 + c_\alpha^2 \|\nabla \psi_0\|_2^2 + 2\alpha c_\alpha \|\psi_\alpha\|_2 \|\nabla \psi_0\|_2 \\ &= \alpha^2 \|\psi_\alpha\|_2^2 + c_\alpha^2 S_4 + 2\alpha c_\alpha \|\psi_\alpha\|_2 S_4^{1/2} \\ &\leq C c_\alpha^2 \alpha^{3/2} + c_\alpha^2 S_4 + 2C^{1/2} c_\alpha^2 \alpha^{3/4} S_4^{1/2}. \end{aligned}$$

Since  $\lim_{\alpha \downarrow 0} c_\alpha = 1$ , we can combine this with (5.81) to arrive at

$$(5.83) \quad \limsup_{\alpha \downarrow 0} \|\nabla \psi_{\alpha,\beta}\|_2^2 \leq S_4.$$

Finally we note that  $\mu_4 \geq S_4$ , giving

$$(5.84) \quad \limsup_{\delta \downarrow 0} (2\delta)^{-1/2} \chi(1-\delta) \leq \mu_4.$$

5. We complete this section by proving that  $\mu_d > 0$  for  $2 \leq d \leq 4$ .

LEMMA 15.

$$(5.85) \quad \mu_2 \geq 1/4S_{2,4}^4 = 27\pi/16, \quad \mu_3 \geq S_3 = 3(\pi/2)^{4/3}, \quad \mu_4 = S_4 = 4\pi\sqrt{6}/3.$$

*Proof.* (1) Rewrite (5.5) as follows:

$$(5.86) \quad \|\nabla f\|_2^2 \geq S_{2,4}\|f\|_4^2 - \|f\|_2^2.$$

Using the scaling in (5.1), (5.2) with  $p > 0$  arbitrary and  $q = 1$ , we have

$$(5.87) \quad \|\nabla f\|_2^2 \geq S_{2,4}p\|f\|_4^2 - p^2\|f\|_2^2.$$

Putting  $\|f\|_2 = \|f\|_4 = 1$  and optimizing over  $p$ , we get the bound for  $\mu_2$ .

(2) The bound for  $\mu_3$  follows from (1.14) via (5.3), (5.4).

(3) The identity for  $\mu_4$  was already proved in Step 4. □

5.6. *Proof of Theorem 5(i).* This section has three parts:

(I) Proof of  $0 < \nu_d < \infty$ .

(II) Proof of  $\Sigma \neq \emptyset$ .

(III) Proof of (1.16).

(I) Proof of  $0 < \nu_d < \infty$ . The upper bound is easily deduced from (1.15) by substituting a test function. The lower bound comes from the following.

*Remark 1.* For  $d \geq 5$

$$(5.88) \quad \nu_d \geq 2^{(d-2)/d} S_d,$$

where  $S_d$  is the sharp constant in the Sobolev inequality given by (5.4).

*Proof.* Since  $e^{-x} - 1 + x \leq \frac{1}{2}x^\alpha$  for  $x \geq 0$  and  $1 \leq \alpha \leq 2$ , it follows that (recall (1.15))

$$(5.89) \quad \nu_d \geq \inf\{\|\nabla \psi\|_2^2 : \int \frac{1}{2}\psi^{2\alpha} \geq 1\} = 2^{1/\alpha} \inf\{\|\nabla \psi\|_2^2 : \|\psi\|_{2\alpha} \geq 1\}.$$

Pick  $\alpha = d/(d-2)$ . Then (5.88) follows from (5.89) via (5.3), (5.4). □

(II) Proof of  $\Sigma \neq \emptyset$ .

1. We begin with a tail estimate.

LEMMA 16. *Let  $d \geq 3$  and let  $\psi \in D^1(\mathbb{R}^d)$  be RSNI. Then*

$$(5.90) \quad 0 \leq \psi(r) \leq \left( \frac{C}{S_d} \right)^{1/2} \omega_d^{-(d-2)/2d} r^{-(d-2)/2}$$

with  $\omega_d = |B_1(0)|$ ,  $r = |x|$  and  $C = \|\nabla\psi\|_2^2$ .

*Proof.* By the Sobolev inequality (5.3), (5.4), we have

$$(5.91) \quad \begin{aligned} \|\nabla\psi\|_2^2 &\geq S_d \|\psi\|_{2d/(d-2)}^2 \geq S_d \|\psi 1_{B_r(0)}\|_{2d/(d-2)}^2 \\ &\geq S_d \psi(r)^2 |B_r(0)|^{(d-2)/d} = S_d \psi(r)^2 \omega_d^{(d-2)/d} r^{d-2}. \end{aligned} \quad \square$$

2. Let  $(\psi_j)$  be a minimising sequence for the variational problem in (1.15). We may assume that  $\psi_j$  is RSNI (recall Section 5.1), and that  $\|\nabla\psi_j\|_2^2 \downarrow \nu_d$ . We can extract a subsequence, again denoted by  $(\psi_j)$ , such that  $\psi_j \rightarrow \psi^*$  weakly in  $D^1(\mathbb{R}^d)$  and  $\psi_j \rightarrow \psi^*$  almost everywhere as  $j \rightarrow \infty$ . It follows that  $\psi^*$  is RSNI too. Moreover,  $\nu_d \geq \|\nabla\psi^*\|_2^2$ . It therefore suffices to show that  $\psi^*$  satisfies the constraint in (1.15), since this implies that  $\nu_d \leq \|\nabla\psi^*\|_2^2$ , and hence that  $\psi^*$  is a minimiser.

3. Let  $\varepsilon > 0$  be arbitrary. Since  $d \geq 5$ , we have  $0 \leq e^{-x} - 1 + x \leq x^{d/(d-2)}$  for  $x \geq 0$ . Hence (5.3) and (5.4) give

$$(5.92) \quad 0 \leq \int (e^{-\psi^{*2}} - 1 + \psi^{*2}) \leq \int \psi^{*2d/(d-2)} \leq \left( \frac{\nu_d}{S_d} \right)^{(d-2)/d}$$

and so there exists an  $R_1(\varepsilon)$  such that

$$(5.93) \quad 0 \leq \int_{B_{R_1(\varepsilon)}^c} (e^{-\psi^{*2}} - 1 + \psi^{*2}) \leq \varepsilon.$$

Let  $C = \sup_j \|\nabla\psi_j\|_2^2 < \infty$  and define  $R_2(\varepsilon)$  by

$$(5.94) \quad \left( \frac{C}{S_d} \right)^2 \omega_d^{-(d-4)/d} \frac{d}{2(d-4)} R_2(\varepsilon)^{-(d-4)} = \varepsilon.$$

Then with the help of Lemma 16 we obtain

$$(5.95) \quad \int_{B_{R_2(\varepsilon)}^c} (e^{-\psi_j^2} - 1 + \psi_j^2) \leq \int_{B_{R_2(\varepsilon)}^c} \frac{1}{2} \psi_j^4 \leq \left( \frac{C}{S_d} \right)^2 \omega_d^{-2(d-2)/d} \frac{1}{2} \int_{B_{R_2(\varepsilon)}^c} |x|^{4-2d} dx = \varepsilon.$$

4. Put  $R(\varepsilon) = \max\{R_1(\varepsilon), R_2(\varepsilon)\}$ . Since  $\int(e^{-\psi_j^2} - 1 + \psi_j^2) = 1$  for all  $j$ , we get from (5.95) that

$$(5.96) \quad 1 - \varepsilon \leq \int_{B_{R(\varepsilon)}} (e^{-\psi_j^2} - 1 + \psi_j^2) \leq 1.$$

Moreover, by Lemma 16,

$$(5.97) \quad 0 \leq e^{-\psi_j^2} - 1 + \psi_j^2 \leq \psi_j^2 \leq \frac{C}{S_d} \omega_d^{-(d-2)/d} r^{-(d-2)},$$

where the right-hand side is integrable on  $B_{R(\varepsilon)}$ . Since  $\psi_j \rightarrow \psi^*$  almost everywhere as  $j \rightarrow \infty$ , we therefore have by the dominated convergence theorem and (5.96) that

$$(5.98) \quad 1 - \varepsilon \leq \int_{B_{R(\varepsilon)}} (e^{-\psi^{*2}} - 1 + \psi^{*2}) \leq 1.$$

Combining (5.93) and (5.98), we obtain

$$(5.99) \quad 1 - \varepsilon \leq \int (e^{-\psi^{*2}} - 1 + \psi^{*2}) \leq 1 + \varepsilon.$$

Since  $\varepsilon$  was arbitrary, we conclude that  $\psi^*$  satisfies the constraint in (1.15). By the same argument as in the proof of Lemma 11, we get that  $\psi^*$  is strictly decreasing in the radial component.

### (III) Proof of (1.16).

1. We begin with the lower bound. Let  $\psi$  be any minimiser of the variational problem (1.15) centered at 0. By the results of Section 5b in Berestycki and Lions [1], there exists a Lagrange multiplier  $\lambda_d > 0$  such that

$$(5.100) \quad (r^{d-1}\psi')' = -\lambda_d r^{d-1}\psi(1 - e^{-\psi^2}),$$

which is the Euler-Lagrange equation in the radial form. By the results of Section 5c in the same paper, we have  $\psi \in C^2(\mathbb{R}^d)$ . Because  $\psi$  is radially symmetric and centered at 0, it follows that  $\psi(0) < \infty$  and  $\psi'(0) = 0$ . Hence  $\psi \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ , and we already know that  $\nabla\psi \in L^2(\mathbb{R}^d)$  and  $\lambda_d(e^{-\psi^2} - 1 + \psi^2) \in L^1(\mathbb{R}^d)$ . It therefore follows from Pohozaev's identity (see Proposition 1 in the same paper) that

$$(5.101) \quad \|\nabla\psi\|_2^2 = \frac{d}{d-2} \lambda_d \int (e^{-\psi^2} - 1 + \psi^2) = \frac{d}{d-2} \lambda_d.$$

Hence the Lagrange multiplier can be identified as

$$(5.102) \quad \lambda_d = \frac{d-2}{d} \nu_d.$$

Multiply both sides of (5.100) by  $\psi$ , integrate over  $r \in [0, \infty)$  and use integration by parts, to get

$$(5.103) \quad \|\nabla\psi\|_2^2 = \lambda_d \int \psi^2(1 - e^{-\psi^2}).$$

Combining this with (5.101), we obtain

$$(5.104) \quad \int \psi^2(1 - e^{-\psi^2}) = \frac{d}{d-2},$$

which obviously implies that  $\|\psi\|_2^2 > d/(d-2)$ .

2. The upper bound is more laborious. We first consider the case  $d \geq 6$ . Then

$$(5.105) \quad \psi(1 - e^{-\psi^2}) \leq \psi^{2d/(d-2)},$$

and hence, with the help of (5.100) and the Sobolev inequality in (5.3), we may estimate

$$(5.106) \quad \begin{aligned} -r^{d-1}\psi'(r) &\leq \lambda_d \int_0^r s^{d-1}\psi(s)^{2d/(d-2)} ds \leq \frac{\lambda_d}{d\omega_d} \int_{\mathbb{R}^d} \psi^{2d/(d-2)} \\ &\leq \frac{\lambda_d}{d\omega_d} \left( \frac{\|\nabla\psi\|_2^2}{S_d} \right)^{d/(d-2)} = \frac{\lambda_d}{d\omega_d} \left( \frac{\nu_d}{S_d} \right)^{d/(d-2)}. \end{aligned}$$

Multiplying both sides of (5.106) by  $r^{1-d}$  and integrating over  $[r, \infty)$ , we get

$$(5.107) \quad \psi(r) \leq Ar^{-(d-2)} \text{ with } A = \frac{\lambda_d}{d(d-2)\omega_d} \left( \frac{\nu_d}{S_d} \right)^{d/(d-2)}.$$

Next, by (5.107) and the Sobolev inequality,

$$(5.108) \quad \begin{aligned} \int \psi^2 &= \int_{\{\psi \geq 1\}} \psi^2 + \int_{\{\psi < 1\}} \psi^2 \leq \int_{\{\psi \geq 1\}} \psi^{2d/(d-2)} + \int_{\{\psi < 1\}} \min\{\psi^2, 1\} \\ &\leq \left( \frac{\nu_d}{S_d} \right)^{d/(d-2)} + d\omega_d \int_0^\infty \min\{A^2 r^{4-2d}, 1\} r^{d-1} dr, \end{aligned}$$

which is finite because  $d \geq 6$ .

3. We note that (5.105) fails for  $d = 5$ . But  $\psi(1 - e^{-\psi^2}) \leq \psi^3$ , and so because  $\psi$  is RSNI we have, by the Sobolev inequality and (5.100),

$$(5.109) \quad -r^4\psi'(r) \leq \lambda_5 \int_0^r s^4\psi(s)^3 ds \leq \psi(r)^{-1/3} \frac{\lambda_5}{5\omega_5} \int \psi^{10/3} \leq \psi(r)^{-1/3} \frac{\lambda_5}{5\omega_5} \left( \frac{\nu_5}{S_5} \right)^{5/3}.$$

Hence  $(\psi(r)^{4/3})' \geq -C_1 r^{-4}$ . Integrating this inequality over  $[r, \infty)$ , we find  $\psi(r) \leq C_2 r^{-9/4}$ . Returning to (5.100) once more, we have by Hölder's inequality,

$$(5.110) \quad \begin{aligned} -r^4\psi'(r) &\leq \lambda_5 \int_0^r s^4\psi(s)^3 ds \\ &\leq \lambda_5 \left( \int_0^r s^4\psi(s)^{10/3} ds \right)^{4/5} \left( \int_0^r s^4\psi(s)^{5/3} ds \right)^{1/5} \leq C_3 r^{1/4}, \end{aligned}$$

where we use the Sobolev inequality for the first integral and our bound on  $\psi(r)$  for the second integral. Multiplying both sides of (5.110) by  $r^{-4}$  and integrating over  $[r, \infty)$ , we arrive at  $\psi(r) \leq C_4 r^{-11/4}$ . But this bound is integrable on the set where  $C_4 r^{-11/4} < 1$ , and so we obtain that  $\psi \in L^2(\mathbb{R}^5)$  via an estimate similar to (5.108).

**5.7. Proof of Theorem 5(ii).** By Lemma 12 and (5.1), (5.2) with  $p = (1-u)^{1/d}$  and  $q = 1$ ,

$$(5.111) \quad \chi(u) = \tilde{\chi}(u) = (1-u)^{(d-2)/d} \inf\{\|\nabla\psi\|_2^2 : \|\psi\|_2^2 \leq (1-u)^{-1}, \int(e^{-\psi^2} - 1 + \psi^2) = 1\}.$$

1. Let  $u \in (0, u_d^*)$ . There exists a RSNI (modulo shifts) minimising sequence  $(\psi_j)$  of the variational problem for  $\chi(u)$  such that  $\|\nabla\psi_j\|_2^2 \rightarrow \chi(u)$  and  $\psi_j \rightarrow \psi^*$  weakly in  $H^1(\mathbb{R}^d)$  as  $j \rightarrow \infty$ . By extracting a subsequence, again denoted by  $(\psi_j)$ , we also have  $\psi_j \rightarrow \psi^*$  almost everywhere as  $j \rightarrow \infty$ . Suppose that  $\|\psi^*\|_2^2 < (1-u)^{-1}$ . Then we can do smooth perturbations of  $\psi^*$  inside the class  $\{\psi \in H^1(\mathbb{R}^d) : \int(e^{-\psi^2} - 1 + \psi^2) = 1\}$  to conclude that  $\psi^*$  must be an element of  $\Sigma^*$ , the set of local minimisers of the variational problem in (1.15). But by the definition of  $u_d^*$  we have  $\|\psi^*\|_2^2 \geq (1-u_d^*)^{-1} \geq (1-u)^{-1}$ , which is a contradiction. Therefore  $\|\psi^*\|_2^2 = (1-u)^{-1}$ , and so  $\psi^*$  is a minimiser of  $\chi(u)$ . Clearly,  $\psi^*$  is RSNI (modulo shifts). By Lemma 11 it must be strictly positive and strictly decreasing in the radial component.

2. Let  $u \in (u_d^+, 1)$ . Suppose that the variational problem for  $\chi(u)$  has a minimiser  $\psi^*$ . Then  $\phi^*(\cdot) = \psi^*(\cdot(1-u)^{1/d})$  is a minimiser of the variational problem in (5.111). Since  $u > u_d^+ \geq u_d^-$ , it follows from (1.18) that  $\|\nabla\phi^*\|_2^2 = \nu_d$ . Hence  $\phi^*$  is also a minimiser of (1.15). But all such minimisers have a squared  $L^2$ -norm that is bounded above by  $(1-u_d^+)^{-1}$ . This is a contradiction because  $\|\phi^*\|_2^2 = (1-u)^{-1}\|\psi^*\|_2^2 = (1-u)^{-1} > (1-u_d^+)^{-1}$ . Hence  $\chi(u)$  has no minimiser. The last claim in Theorem 5(ii) is obvious.

### 5.8. Proof of Theorem 5(iii).

1. By dropping the constraint  $\|\psi\|_2^2 \leq (1-u)^{-1}$  from (5.111) and recalling (1.15), we see that  $\chi(u) \geq (1-u)^{(d-2)/d}\nu_d$ , which proves the lower bound in (1.18). The upper bound is proved via an argument similar to Step ii in the proof of Lemma 12. Let  $u \in (u_d^-, 1)$ . Then, by Theorem 5(i), there exists a  $\psi^* \in \Sigma$  satisfying  $\|\psi^*\|_2^2 < (1-u)^{-1}$ . For  $n \in \mathbb{N}$ , define  $\psi_{n,u}^*$  by

$$(5.112) \quad \psi_{n,u}^{*2} = \psi^{*2} + \left( \frac{1}{1-u} - \|\psi^*\|_2^2 \right) p_n,$$

with  $p_n$  as in (5.21). Then  $\|\psi_{n,u}^*\|_2^2 = (1-u)^{-1}$  for all  $n$ . Moreover, since  $x \mapsto e^{-x} - 1 + x$  is increasing on  $[0, \infty)$ , we have

$$(5.113) \quad \int(e^{-\psi_{n,u}^{*2}} - 1 + \psi_{n,u}^{*2}) \geq \int(e^{-\psi^{*2}} - 1 + \psi^{*2}) = 1.$$

Therefore, by (5.111) and the convexity inequality for gradients,

$$\begin{aligned}
 (5.114) \quad (1-u)^{-(d-2)/d} \chi(u) &\leq \|\nabla \psi_{n,u}^*\|_2^2 \\
 &\leq \|\nabla \psi^*\|_2^2 + \left( \frac{1}{1-u} - \|\psi^*\|_2^2 \right) \|\sqrt{\nabla p_n}\|_2^2 \\
 &\leq \nu_d + (1-u)^{-1} \frac{2d}{n^2}.
 \end{aligned}$$

Let  $n \rightarrow \infty$  to get the upper bound in (1.18). The case  $u = u_d^-$  follows by continuity of  $\chi$ .

2. It is immediate from (5.111) that  $u \mapsto (1-u)^{-(d-2)/d} \chi(u)$  is nonincreasing on  $(0, 1)$ . Strict monotonicity on  $(0, u_d^*)$  follows from the existence of a minimiser via the same type of argument as in Step 2 in Section 5.5. Finally, the fact that  $u \mapsto (1-u)^{-(d-2)/d} \chi(u) > \nu_d$  for all  $u \in (u_d^*, u_d^-)$  needs to be proved only when  $u_d^* < u_d^-$ . In that case  $\Sigma^* \setminus \Sigma \neq \emptyset$ . But clearly  $\|\nabla \psi\|_2^2 > \nu_d$  for all  $\psi \in \Sigma^* \setminus \Sigma$ , and so weak convergence to any such  $\psi$  will not reach the minimal value  $\nu_d$ .

We conclude by settling an old debt: to prove the strict monotonicity in Theorem 3(iii). By Theorems 4(i) and 5(ii), (5.43) has a minimiser for  $2 \leq d \leq 4$ ,  $u \in (0, 1)$ , and for  $d \geq 5$ ,  $u \in (0, u_d^*]$ , in which case the claim follows via the same type of argument as in Step 2 in Section 5.5. On the other hand, for  $d \geq 5$ ,  $u \in (u_d^*, 1)$ , we can appeal to Theorem 5(iii), which easily gives the claim because  $u_d^* \geq 2/d$ .

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#### REFERENCES

- [1] H. BERESTYCKI and P. L. LIONS, Nonlinear scalar field equations, I. Existence of a ground state, II. Existence of infinitely many solutions, *Arch. Rat. Mech. Anal.* **82** (1983), 313–345 and 347–375.
- [2] M. VAN DEN BERG and E. BOLTHAUSEN, Asymptotics of the generating function for the volume of the Wiener sausage, *Probab. Theory Related Fields* **99** (1994), 389–397.
- [3] M. VAN DEN BERG and E. B. DAVIES, Heat flow out of regions in  $\mathbb{R}^m$ , *Math. Z.* **202** (1989), 463–482.
- [4] M. VAN DEN BERG and B. TÓTH, Exponential estimates for the Wiener sausage, *Probab. Theory Related Fields* **88** (1991), 249–259.
- [5] E. BOLTHAUSEN, Markov process large deviations in the  $\tau$ -topology, *Stoch. Process. and Appl.* **25** (1987), 95–108.
- [6] ———, On the volume of the Wiener sausage, *Ann. Probab.* **18** (1990), 1576–1582.

- [7] E. BOLTHAUSEN, Localization of a two-dimensional random walk with an attractive path interaction, *Ann. Probab.* **22** (1994), 875–918.
- [8] H. BREZIS and E. H. LIEB, A relation between pointwise convergence of functions and convergence of functionals, *Proc. A. M. S.* **88** (1983), 486–490.
- [9] J. E. BROTHERS and W. P. ZIEMER, Minimal rearrangements of Sobolev functions, *J. reine angew. Math.* **384** (1988), 153–179.
- [10] W. BRYC, Large deviations by the asymptotic value method, in: *Diffusion Processes and Related Problems in Analysis* (M. Pinsky, ed.), Vol. 1, pp. 447–472, Birkhäuser, Boston, 1990.
- [11] I. Csiszár, Information-type measures of difference of probability distributions and indirect observations, *Studia Sci. Math. Hungar.* **2** (1967), 299–318.
- [12] M. D. DONSKER and S. R. S. VARADHAN, Asymptotics for the Wiener sausage, *Comm. Pure Appl. Math.* **28** (1975), 525–565; Erratum, *ibid.* **28** (1975), 677.
- [13] ———, Asymptotic evaluation of certain Markov process expectations for large time, *Comm. Pure Appl. Math.* (I) **28** (1975), 1–47, (II) **28** (1975), 279–301, (III) **29** (1976), 389–461, (IV) **36** (1983), 183–212.
- [14] W. G. FARIS, Weak Lebesgue spaces and quantum mechanical binding, *Duke Math. J.* **43** (1976), 365–373.
- [15] R. K. GETTOOR, Some asymptotic formulas involving capacity, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **4** (1965), 248–252.
- [16] J.-F. LE GALL, Sur la saucisse de Wiener et les points multiples du mouvement brownien, *Ann. Probab.* **14** (1986), 1219–1244.
- [17] ———, Sur une conjecture de M. Kac, *Probab. Theory Related Fields* **78** (1988), 389–402.
- [18] ———, Fluctuation results for the Wiener sausage, *Ann. Probab.* **16** (1988), 991–1018.
- [19] E. H. LIEB and M. LOSS, *Analysis, Grad. Stud. in Math.* **14**, A. M. S., Providence, RI, 1997.
- [20] T. POVEL, Confinement of Brownian motion among Poissonian obstacles in  $\mathbb{R}^d$ ,  $d \geq 3$ , *Probab. Theory Related Fields* **114** (1999), 177–205.
- [21] F. SPITZER, Electrostatic capacity, heat flow and Brownian motion, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **3** (1964), 110–121.
- [22] A.-S. SZNITMAN, Long time asymptotics for the shrinking Wiener sausage, *Comm. Pure Appl. Math.* **43** (1990), 809–820.
- [23] ———, On the confinement property of two-dimensional Brownian motion among Poissonian obstacles, *Comm. Pure Appl. Math.* **44** (1991), 1137–1170.
- [24] ———, *Brownian Motion, Obstacles and Random Media*, Springer-Verlag, Berlin, 1998.
- [25] M. TALAGRAND, Concentration of measure and isoperimetric inequalities in product spaces, *IHES Publ. Math.* **81** (1995), 73–205.

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